ISRAEL JOURNAL OF MATHEMATICS 95 (1996), 61–114

FURTHER CARDINAL ARITHMETIC

BY

SAHARON SHELAH*

Institute of Mathematics
The Hebrew University of Jerusalem, Jerusalem, Israel
and
Department of Mathematics
Rutgers University, New Brunswick, NJ, USA

ABSTRACT

We continue the investigations in the author's book on cardinal arithmetic, assuming some knowledge of it. We deal with the cofinality of $(\mathcal{S}_{\leq\aleph_0}(\kappa), \subseteq)$ for κ real valued measurable (Section 3), densities of box products (Section 5,3), prove the equality $\mathrm{cov}(\lambda, \lambda, \theta^+, 2) = \mathrm{pp}(\lambda)$ in more cases even when $\mathrm{cf}(\lambda) = \aleph_0$ (Section 1), deal with bounds of $\mathrm{pp}(\lambda)$ for λ limit of inaccessible (Section 4) and give proofs to various claims I was sure I had already written but did not find (Section 6).

Annotated Contents

1. Equivalence of two covering properties 63

[We try to characterize when, say, λ has few countable subsets; for a given $\theta \in (\aleph_0, \lambda)$, we try to translate to expressions with pcf's the cardinal

$$\mathrm{Min}\Big\{|\mathcal{P}|\colon \mathcal{P} \subseteq \mathcal{S}_{<\mu}(\lambda) \text{ and every } a \in \mathcal{S}_{\leq\theta}(\lambda) \text{ is } \bigcup_{n<\omega} a_n, \text{ such that every}$$
$$b \in \bigcup_n \mathcal{S}_{\leq\aleph_0}(a_n) \text{ is included in a member of } \mathcal{P}\Big\}.$$

This continues and improves [Sh410,§6].]

2. Equality relevant to weak diamond 70

* Done mainly 1–4/1991.
I thank Alice Leonhardt for typing and retyping so beautifully and accurately.
Partially supported by the Basic Research Fund, Israel Academy of Sciences.
Publication number 430.
Received January 23, 1992 and in final revised form April 22, 1996

[We show that if $\mu > \lambda \geq \kappa$, $\theta = \mathrm{cov}(\mu, \lambda^+, \lambda^+, \kappa)$ and $\mathrm{cov}(\lambda, \kappa, \kappa, 2) \leq \mu$ (or $\leq \theta$), then $\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) = \mathrm{cov}(\theta, \kappa, \kappa, 2)$. This is used in [Sh-f, Appendix,§1] to clarify the conditions for the holding of versions of the weak diamond.]

3. Cofinality of $\mathcal{S}_{\leq \aleph_0}(\kappa)$ for κ real valued measurable and trees 72
[Dealing with partition theorems on trees, Rubin–Shelah [RuSh117] arrive at the statement: $\lambda > \kappa > \aleph_0$ are regular, $a_\alpha \in \mathcal{S}_{<\kappa}(\mu)$, $\mu < \lambda$; can we find unbounded $W \subseteq \lambda$ such that $|\bigcup_{\alpha \in W} a_\alpha| < \kappa$? Of course, $\bigwedge_{\alpha < \lambda} \mathrm{cov}(\alpha, \kappa, \kappa, 2) < \lambda$ suffice, but is it necessary? By 3.1, yes. Then we answer a problem of Fremlin: e.g. if κ is a real valued measurable cardinal then the cofinality of $(\mathcal{S}_{\leq \aleph_0}(\kappa), \subseteq)$ is κ. Lastly we return to the problem of the existence of trees with many branches (3.3, 3.4).]

4. Bounds for $\mathrm{pp}_{\Gamma(\aleph_1)}$ for limits of inaccessibles 79
[Unfortunately, our results need an assumption: $\mathrm{pcf}(\mathfrak{a})$ does not have an inaccessible accumulation point ($|\mathfrak{a}| < \mathrm{Min}\, \mathfrak{a}$, $\mathfrak{a} \subseteq \mathrm{Reg}$, of course). Our main conclusion (4.3) is that e.g. if $\langle \lambda_\zeta : \zeta < \omega_4 \rangle$ is the list of the first $\aleph_4$ inaccessibles then $\mathrm{pp}_{\Gamma(\aleph_1)}\left(\bigcup_{\zeta < \omega_1} \lambda_\zeta\right) < \bigcup_{\zeta < \omega_4} \lambda_\zeta$. This does not follow from the proof of $\mathrm{pp}\, \aleph_\omega < \aleph_{\omega_4}$ [Sh400,§2], nor do we make our life easier by assuming "$\bigcup_{\zeta < \omega_1} \lambda_\zeta$ is strong limit". We indeed in the end quote a variant of [Sh400,§2] (= [Sh410,3.5]). But the main point now is to arrive at the starting point there: show that for $\delta < \omega_4$, $\mathrm{cf}\, \delta = \aleph_2$, for some club C of δ, $\sup \mathrm{pcf}_{\aleph_2\text{-complete}}(\{\lambda_\zeta : \zeta \in C\})$ is $\leq \lambda_\delta$. This is provided by 4.2.]

5. Densities of box products* 85
[The behavior of the Tichonov product of topological spaces on densities is quite well understood for ${}^\mu 2$: it is $\mathrm{Min}\{\lambda : 2^\lambda \geq \mu\}$; but less so for the generalization to box products. Let $\mathcal{T}_{\mu,\theta,\kappa}$ be the space with set of points ${}^\mu\theta$, and basis $\{[f] : f$ a partial function from μ to θ of cardinality $< \kappa\}$, where $[f] = \{g \in {}^\mu\theta : f \subseteq g\}$. If $\theta \leq \lambda = \lambda^{<\kappa}$, $2^\lambda \geq \mu$ the situation is similar to the Tichonov product. Now the characteristic unclear case is μ strong limit singular of cofinality $< \kappa$, $\theta = 2$, $2^\mu > \mu^+$. We prove that the density is "usually" large (2^μ), i.e. the failure quite limits the cardinal arithmetic involved (we can prove directly consistency results but what we do seems more informative).]

6. Odds and ends 90

References 112

Notation: Let $J_\lambda[\mathfrak{a}]$ be $\{\mathfrak{b} \subseteq \mathfrak{a} : \lambda \notin \mathrm{pcf}(\mathfrak{b})\}$, equivalently $J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda[\mathfrak{a}]$. See more in [Sh513], [Sh589].

* There is a paper in preparation on independence results by Gitik and Shelah.

1. Equivalence of Two Covering Properties

1.1 CLAIM: *If* $\mathrm{pp}\,\lambda = \lambda^+$, $\lambda > \mathrm{cf}(\lambda) = \kappa > \aleph_0$ *then* $\mathrm{cov}(\lambda, \lambda, \kappa^+, 2) = \lambda^+$.

Proof: Let $\chi = \beth_3(\lambda)^+$; choose $\langle \mathfrak{B}_\zeta\colon \zeta < \lambda^+\rangle$ increasing continuous, such that $\mathfrak{B}_\zeta \prec (H(\chi), \in, <^*_\chi)$, $\lambda + 1 \subseteq \mathfrak{B}_\zeta$, $\|\mathfrak{B}_\zeta\| = \lambda$ and $\langle \mathfrak{B}_\xi\colon \xi \le \zeta\rangle \in \mathfrak{B}_{\zeta+1}$. Let $\mathfrak{B} =: \bigcup_{\zeta<\lambda^+} \mathfrak{B}_\zeta$ and $\mathcal{P} =: \mathcal{S}_{<\lambda}(\lambda) \cap \mathfrak{B}$. Let $a \in \mathcal{S}_{\le\kappa}(\lambda)$; it suffices to prove $(\exists A \in \mathcal{P})[a \subseteq A]$. Let f_ξ be the $<^*_\chi$-first $f \in \prod(\mathrm{Reg} \cap \lambda)$ such that $(\forall g)[g \in \prod(\mathrm{Reg} \cap \lambda) \,\&\, g \in \mathfrak{B}_\zeta \Rightarrow g < f \bmod J^{bd}_\lambda]$, such f exists as $\prod(\mathrm{Reg} \cap \lambda)/J^{bd}_\lambda$ is λ^+-directed.

By [Sh420, 1.5, 1.2] we can find $\langle C_\alpha\colon \alpha < \lambda^+\rangle$ such that: C_α is a closed subset of α, $\mathrm{otp}\, C_\alpha \le \kappa^+$, $[\beta \in \mathrm{nacc}\, C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta]$ and $S =: \{\delta < \lambda^+\colon \mathrm{cf}(\delta) = \kappa^+$ and $\delta = \sup C_\delta\}$ is stationary.

Without loss of generality $\bar{C} \in \mathfrak{B}_0$.

Now we define for every $\alpha < \lambda^+$ elementary submodels N^0_α, N^1_α of $\mathfrak{B}$:

N^0_α is the Skolem Hull of $\{f_\zeta\colon \zeta \in C_\alpha\} \cup \{i\colon i \le \kappa\}$ and N^1_α is the Skolem Hull of $a \cup \{f_\zeta\colon \zeta \in C_\alpha\} \cup \{i : i \le \kappa\}$, both in $(H(\chi), \in, <^*_\chi)$.

Clearly:

(a) $N^0_\alpha \subseteq N^1_\alpha \subseteq \mathfrak{B}_\alpha \subseteq \mathfrak{B}$ [why? as $f_\zeta \in \mathfrak{B}_{\zeta+1}$ because $\mathfrak{B}_\zeta \in \mathfrak{B}_{\zeta+1}$],

(b) $\|N^\ell_\alpha\| \le \kappa + \|C_\alpha\|$,

(c) $N^0_\alpha \in \mathfrak{B}_{\alpha+1}$.

[Why? As $\alpha \subseteq \mathfrak{B}_\alpha$ (you can prove it by induction on α) clearly $\alpha \in \mathfrak{B}_{\alpha+1}$, but $\bar{C} \in \mathfrak{B}_0 \subseteq \mathfrak{B}_{\alpha+1}$; hence $C_\alpha \in \mathfrak{B}_{\alpha+1}$, also $\langle \mathfrak{B}_\gamma\colon \gamma \le \alpha\rangle \in \mathfrak{B}_{\alpha+1}$ hence $\langle f_\gamma\colon \gamma \le \alpha\rangle \in \mathfrak{B}_{\alpha+1}$, hence $\langle f_\gamma\colon \gamma \in C_\alpha\rangle \in \mathfrak{B}_{\alpha+1}$. Now $N^0_\alpha \subseteq \mathfrak{B}_\alpha \in \mathfrak{B}_{\alpha+1}$ and the Skolem Hull can be computed in $\mathfrak{B}_{\alpha+1}$.]

(d) for each α with $\kappa^+ > \mathrm{otp}(C_\alpha)$, for some $\gamma_\alpha < \lambda^+$, letting $\mathfrak{a}_\alpha =: N^0_\alpha \cap \mathrm{Reg} \cap \lambda \backslash \kappa^{++}$ clearly $\mathrm{Ch}_\alpha \in \prod \mathfrak{a}_\alpha$ where $\mathrm{Ch}_\alpha(\theta) =: \sup(\theta \cap N^1_\alpha)$, and we have: $\mathrm{Ch}_\alpha < f_{\gamma_\alpha} \restriction \mathfrak{a}_\alpha \bmod J^{bd}_{\mathfrak{a}_\alpha}$.

[Why? $\mathfrak{a}_\alpha \in \mathfrak{B}_{\alpha+1}$ as $N^0_\alpha \in \mathfrak{B}_{\alpha+1}$, and $\prod \mathfrak{a}_\alpha / J^{bd}_{\mathfrak{a}_\alpha}$ is λ^+-directed (trivially) and has cofinality $\le \max \mathrm{pcf}_{J^{bd}_{\mathfrak{a}_\alpha}}(\mathfrak{a}_\alpha) \le \mathrm{pp}(\lambda) = \lambda^+$, so there is $\langle f^{\mathfrak{a}_\alpha}_\beta\colon \beta < \lambda^+\rangle$, $<_{J^{bd}_{\mathfrak{a}_\alpha}}$-increasing cofinal sequence in $\prod \mathfrak{a}_\alpha$, so without loss of generality $\langle f^{\mathfrak{a}_\alpha}_\beta\colon \beta < \lambda^+\rangle \in \mathfrak{B}_{\alpha+1}$; also by the "cofinal" above, for some $\beta \in (\alpha, \lambda^+)$, $\mathrm{Ch}_\alpha < f^{\mathfrak{a}_\alpha}_\beta \bmod J^{bd}_{\mathfrak{a}_\alpha}$. We can use the minimal β, now obviously $\beta \in \mathfrak{B}_{\beta+1}$ so $f^{\mathfrak{a}_\alpha}_\beta \in \mathfrak{B}_{\beta+1}$, hence $f^{\mathfrak{a}_\alpha}_\beta < f_{\beta+2} \bmod J^{bd}_\lambda$. Together $\gamma_\alpha =: \beta + 2$ is as required.]

(d)$^+$ for each α with $\mathrm{otp}(C_\alpha) < \kappa^+$ for some $\gamma_\alpha \in (\alpha, \lambda^+)$, for any $\mu \in \mathrm{Reg} \cap N^0_\alpha$, letting $N^{0,\mu}_\alpha =: \mathrm{Ch}_{\mathfrak{B}_\alpha}(N^0_\alpha \cup \mu)$, $\mathfrak{a}_{\alpha,\mu} = N^{0,\mu}_\alpha \cap \mathrm{Reg} \cap \lambda \setminus \mu^+$ and $\mathrm{Ch}_{\alpha,\mu} \in$

$\Pi\mathfrak{a}_{\alpha,\mu}$ be

$$\mathrm{Ch}_{\alpha,\mu}(\theta) = \begin{cases} \sup(\theta \cap N^1_\alpha) & \text{if } \theta \in N^1_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

we have: $\mathrm{Ch}_\alpha < f_{\gamma_\alpha} \restriction \mathfrak{a}_{\alpha,\mu} \bmod J^{bd}_{\mathfrak{a}_{\alpha,\mu}}$.

[Why? Clearly $\mathrm{Ch}_{\mathfrak{B}_\alpha}(N^0_\alpha \cup \mu) \in \mathfrak{B}_{\alpha+1}$, so $\mathfrak{a}_{\alpha,\mu} \in \mathfrak{B}_{\alpha+1}$, hence there are in $\mathfrak{B}_{\alpha+1}$ elements $\langle \mathfrak{b}_\theta[\mathfrak{a}_{\alpha,\mu}]: \theta \in \mathrm{pcf}(\mathfrak{a}_{\alpha,\mu})\rangle$ and $\langle\langle f^{\mathfrak{a}_{\alpha,\mu},\theta}_\alpha : \alpha < \theta\rangle: \theta \in \mathrm{pcf}(\mathfrak{a}_{\alpha,\mu})\rangle$ as in [Sh 371, 2.6, §1]. So for some $\gamma_{\alpha,\mu} \in (\alpha, \lambda^+)$ we have $\mathrm{Ch}_\alpha \restriction \mathfrak{b}_{\lambda^+}[\mathfrak{a}_{\alpha,\mu}] < f_{\gamma_\alpha}$, so it is enough to prove $\mathfrak{a}_{\alpha,\mu} \setminus \mathfrak{b}_{\lambda^+}[\mathfrak{a}_{\alpha,\mu}]$ is bounded below μ but otherwise $\mathrm{pp}(\lambda) = \lambda^+$ will be contradicted. Let $\gamma_\alpha = \sup\{\gamma_{\alpha,\mu}: \mu \in N^0_\alpha\}$.]

(e) $E^* =: \{\delta < \lambda^+: \alpha < \delta \,\&\, |C_\alpha| \le \kappa \Rightarrow \gamma_\alpha < \delta \text{ and } \delta > \lambda\}$ is a club of λ.

Now as S is stationary, there is $\delta(*) \in S \cap E^*$. Remember $\mathrm{otp}\, C_{\delta(*)} = \kappa^+$.

Let $C_{\delta(*)} = \{\alpha_{\delta(*),\zeta}: \zeta < \kappa^+\}$ (in increasing order).

Let (for any $\zeta < \kappa^+$) M^0_ζ be the Skolem Hull of $\{f_{\alpha_{\delta(*),\xi}}: \xi < \zeta\} \cup \{i : i \le \kappa\}$, and let M^1_ζ be the Skolem Hull of $a \cup \{f_{\alpha_{\delta(*),\xi}}: \xi < \zeta\} \cup \{i : i \le \kappa\}$. Note: for $\zeta < \kappa^+$ non-limit $\{f_{\alpha_{\delta(*),\xi}}: \xi < \zeta\} = \{f_\xi: \xi \in C_{\alpha_{\delta(*),\zeta}}\}$. Clearly $\langle M^0_\zeta: \zeta < \kappa^+\rangle$, $\langle M^1_\zeta: \zeta < \kappa^+\rangle$ are increasing continuous sequences of countable elementary submodels of $\mathfrak{B}$ and $M^0_\zeta \subseteq M^1_\zeta$ and for $\zeta < \kappa^+$ a successor ordinal, $N^\ell_{\alpha_{\delta(*),\zeta}} = M^\ell_\zeta$.

Now for each successor ζ, for some $\epsilon(\zeta) \in (\zeta, \omega_1)$ we have $\gamma_{\alpha_{\delta(*),\zeta}} < \alpha_{\delta(*),\epsilon(\zeta)}$ (by the choice of $\delta(*)$) hence $f_{\gamma_{\alpha_{\delta(*),\zeta}}} < f_{\alpha_{\delta(*),\epsilon(\zeta)}} \bmod J^{bd}_\lambda$ hence $\mathrm{Ch}_{\alpha_{\delta(*),\zeta}} < f_{\alpha_{\delta(*),\epsilon(\zeta)}} \bmod J^{bd}_\lambda$.

Let $E =: \{\delta < \omega_1: \text{for every successor } \zeta < \delta,\ \epsilon(\zeta) < \delta\}$, clearly E is a club of κ^+. Let $\lambda = \sum_{i<\kappa} \lambda_i$, $\lambda_i < \lambda$ singular increasing continuous with i, wlog $\{\lambda_i: i < \kappa\} \subseteq \mathrm{Ch}_{\mathfrak{B}}(\{i: i \le \kappa\} \cup \{\lambda\})$. So for some $\mu_{\zeta,i} < \lambda$, we have:

$$(*)\qquad \begin{aligned} &i < \kappa,\quad \zeta = \xi + 1 < \kappa^+ \,\&\, \theta \in \mathrm{Reg} \cap \lambda \backslash \mu_{\zeta,i} \,\&\, \theta \in N^{0,\lambda_i}_{\alpha_{\delta(*),\zeta}} \cap N^1_{\alpha_{\delta(*),\zeta}} \\ &\qquad \Rightarrow \sup\left(N^1_{\alpha_{\delta(*),\zeta}} \cap \theta\right) < f_{\alpha_{\delta(*),\epsilon(\zeta)}}(\theta) \in \theta \cap N^{0,\lambda_i}_{\alpha_{\delta(*),\zeta+1}}. \end{aligned}$$

So for some limit $i(\zeta) < \kappa^+$ we have $\lambda_{i(\zeta)} = \sup\{\mu_{\zeta,j}: j < i(\zeta)\}$. Now as $\mathrm{cf}\,\lambda \le \kappa^+$ for some $i(*) < \lambda$

$$W =: \{\zeta < \kappa^+: \zeta \text{ successor ordinal and } i(\zeta) = i(*)\}$$

is unbounded in κ^+. So

$$\otimes\qquad \begin{aligned} &\text{if } \xi < \kappa^+,\ \xi \in E,\ \xi = \sup(\xi \cap W) \text{ and } \theta \in M^1_\xi\, \mathrm{Reg} \cap \lambda \cap M^{0,\lambda_{i(*)}}_\xi \backslash \lambda_{i(*)} \\ &\text{then } M^{0,\lambda_i}_\xi \cap \theta \text{ is an unbound subset of } M^1_\xi \cap \theta. \end{aligned}$$

Hence by [Sh400] 5.1A(1), remembering $M^0_{\zeta+1} = N^0_{\alpha_{\delta(*)},\zeta+1}$, we have: $M^1_\xi \subseteq$ Skolem Hull$\left[\bigcup_{\zeta<\xi} N^0_{\zeta+1} \cup \lambda_{i(*)}\right] \subseteq$ Skolem Hull $\left(N^0_{\alpha_{\delta(*)},\xi+1} \cup \lambda_{i(*)}\right)$ whenever $\xi \in E$ is an accumulation point of W. But $a \subseteq M^1_\xi$ and the right side belongs to $\mathfrak{B}$ (as we can take the Skolem Hull in $\mathfrak{B}_{\delta(*)}$). So we have finished. $\blacksquare_{1.1}$

Remark: Alternatively note: $\mathrm{cov}(\lambda, \lambda, \kappa, 2) \leq \mathrm{cov}(\theta, \lambda, \sigma, 2)$ when $\sigma = \mathrm{cf}(\lambda) < \kappa < \lambda$, $\sigma => \aleph_0$, $\theta = \mathrm{pp}_{\Gamma(\kappa,\sigma)}(\lambda)$; remember $\mathrm{cf}(\lambda) < \kappa < \lambda$ & $\mathrm{pp}(\lambda) < \lambda^{+\kappa^+} \Rightarrow \mathrm{pp}_{<\lambda}(\lambda) = \mathrm{pp}(\lambda)$.

1.2 Claim: *For $\lambda > \mu = \mathrm{cf}(\mu) > \theta > \aleph_0$, we have $\lambda(0) \leq \lambda(1) \leq \lambda(2) = \lambda(3)$ and if $\mathrm{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$ they are all equal, where:*

$\lambda(0) =$: *is the minimal κ such that: if $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$, $|\mathfrak{a}| \leq \theta$ then we can find $\langle \mathfrak{a}_\ell : \ell < \omega \rangle$ such that $\mathfrak{a} = \bigcup_{\ell<\omega} \mathfrak{a}_\ell$ and $(\forall \mathfrak{b})\left[\mathfrak{b} \in \mathcal{S}_{\leq\aleph_0}(\mathfrak{a}_n) \Rightarrow \max \mathrm{pcf}(\mathfrak{b}) \leq \kappa\right]$.*

$\lambda(1) =: \mathrm{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{S}_{<\mu}(\lambda)$, *and for every $A \subseteq \lambda, |A| \leq \theta$ there are $A_n \subseteq A$ $(n < \omega)$, $A = \bigcup_{n<\omega} A_n$, $A_n \subseteq A_{n+1}$ such that: for $n < \omega$, every $a \in \mathcal{S}_{\leq\aleph_0}(A_n)$ is a subset of some member of $\mathcal{P}$*$\}$.

$\lambda(2)$ is defined similarly to $\lambda(1)$ as:

$$\mathrm{Min}\Big\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{S}_{<\mu}(\lambda) \text{ and for every } A \in \mathcal{S}_{\leq\theta}(\lambda) \text{ for some } A_n \subseteq A (n < \omega)\ A = \bigcup_{n<\omega} A_n \text{ and for each } n < \omega \text{ for some } \mathcal{P}_n \subseteq \mathcal{P},\ |\mathcal{P}_n| < \mu,\ \sup_{B\in\mathcal{P}_n} |B| < \mu \text{ and every } a \in \mathcal{S}_{\leq\aleph_0}(A_n) \text{ is a subset of some member of } \mathcal{P}_n\Big\}.$$

$\lambda(3)$ *is the minimal κ such that: if $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$, $|\mathfrak{a}| \leq \theta$, then we can find $\langle \mathfrak{a}_\ell : \ell < \omega \rangle$, $\mathfrak{a}_\ell \subseteq \mathfrak{a}_{\ell+1} \subseteq \mathfrak{a} = \bigcup_{\ell<\omega} \mathfrak{a}_\ell$ such that: there is $\{\mathfrak{b}_{\ell,i} : i < i_\ell < \mu\}$, $\mathfrak{b}_{\ell,i} \subseteq \mathfrak{a}_\ell$ such that $\max \mathrm{pcf}\, \mathfrak{b}_{\ell,i} \leq \kappa$ and $(\forall \mathfrak{c})[\mathfrak{c} \subseteq \mathfrak{a}_\ell \& |\mathfrak{c}| \leq \aleph_0 \Rightarrow \bigvee_i \mathfrak{c} \subseteq \mathfrak{b}_{\ell,i}]$; equivalently: $\mathcal{S}_{\leq\aleph_0}(\mathfrak{a}_n)$ is included in the ideal generated by $\{\mathfrak{b}_\sigma[\mathfrak{a}_n] : \sigma \in \mathfrak{d}\}$ for some $\mathfrak{d} \subseteq \kappa^+ \cap \mathrm{pcf}\, \mathfrak{a}_n$ of cardinality $< \mu$.*

1.2A Remark: (1) We can get similar results with more parameters: replacing $\aleph_0$ and/or $\aleph_1$ by higher cardinals.

(2) Of course, by assumptions as in [Sh410, §6] (e.g. $|\operatorname{pcf}\mathfrak{a}| \leq |\mathfrak{a}|$) we get $\lambda(0) = \lambda(3)$. This (i.e. Claim 1.2) will be continued in [Sh513].

Proof:

$\lambda(1) \leq \lambda(2)$: Trivial.

$\lambda(2) \leq \lambda(3)$: Let $\chi = \beth_3(\lambda(3))^+$ and for $\zeta \leq \mu^+$ we choose $\mathfrak{B}_\zeta \prec (H(\chi), \in, <_\chi)$, $\{\lambda, \mu, \theta, \lambda(2), \lambda(3)\} \in \mathfrak{B}_\zeta$, $\|\mathfrak{B}_\zeta\| = \lambda(3)$ and $\lambda(3) \subseteq \mathfrak{B}_\zeta$, $\mathfrak{B}_\zeta$ $(\zeta \leq \mu^+)$ increasing continuous and $\langle \mathfrak{B}_\xi : \xi \leq \zeta \rangle \in \mathfrak{B}_{\zeta+1}$ and let $\mathfrak{B} = \mathfrak{B}_{\mu^+}$. Lastly let $\mathcal{P} = \mathfrak{B} \cap \mathcal{S}_{<\mu}(\lambda)$. Clearly

$(*)_0$ a function $\mathfrak{a} \mapsto \langle \mathfrak{b}_\sigma[\mathfrak{a}] : \sigma \in \operatorname{pcf}\mathfrak{a} \rangle$ as in [Sh371, 2.6] is definable in $(H(\chi), \in, <^*_\chi)$ hence $\mathfrak{B}$ is closed under it.

It suffices to show that $\mathcal{P}$ satisfies the requirements in the definition of $\lambda(2)$.

Let $A \subseteq \lambda$, $|A| \leq \theta$. We choose by induction on $n < \omega$, N^a_n, (for $\ell < \omega$) and N^b_n, f_n such that:

(a) N^a_n, N^b_n are elementary submodels of $(H(\chi), \in, <^*_\chi)$ of cardinality θ,

(b) $f_n \in \prod \mathfrak{a}_n$ where $\mathfrak{a}_n =: N^a_n \cap \operatorname{Reg} \cap \lambda^+ \setminus \mu$, and $f_n(\sigma) > \sup(N^b_n \cap \sigma)$ (for any $\sigma \in \mathfrak{a}_n$),

(c) $\theta + 1 \subseteq N^a_n \subseteq N^b_n \subseteq \mathfrak{B}$,

(d) N^b_n is the Skolem Hull of $\bigcup\{\operatorname{Rang} f_\ell : \ell < n\} \cup A \cup (\theta + 1)$,

(e) N^a_0 is the Skolem Hull of $\theta + 1$ in $(H(\chi), \in, <^*_\chi)$,

(f) N^a_{n+1} is the Skolem Hull of $N^a_n \cup \operatorname{Rang} f_n$,

(g) there are $\mathcal{P}_{n,\ell} \subseteq \mathcal{S}_{<\mu}(\lambda + 1)$ and $A_{n,\ell} \subseteq N^a_n$ (for $l < \omega$) such that:

(α) $|\mathcal{P}_{n,\ell}| < \mu$ and $\mu_{n,\ell} =: \sup_{B \in \mathcal{P}_{n,\ell}} |B| < \mu$ and $\mathcal{P}_{n,\ell} \subseteq \mathcal{P}_{n,\ell+1}$,

(β) $N^a_n = \bigcup_\ell A_{n,\ell}$, $\mathcal{P}_n = \bigcup_{\ell<\omega} \mathcal{P}_{n,\ell} \subseteq \mathfrak{B}$ and $A_{n,\ell} \subseteq A_{n,\ell+1}$,

(γ) for every countable $a \subseteq \lambda \cap A_{n,\ell}$ there is $b \in \mathcal{P}_{n,\ell}$ satisfying $a \subseteq b$,

(δ) $\mathcal{P}_{n,\ell} = \mathcal{S}_{\leq \mu_{n,\ell}}(\lambda + 1) \cap (\text{Skolem Hull of } A_{n,\ell} \cup \mathcal{P}_{n,\ell} \cup (\theta + 1))$.

As in previous proofs, if we succeed to carry out the definition, then $\bigcup_n (N^a_n \cap \lambda) = \bigcup_n N^b_n \cap \lambda$, but the former is $\bigcup_{n,\ell} A_{n,\ell} \cap \lambda$, hence $A \subseteq \bigcup_n \bigcup_\ell A_{n,\ell}$, by (g)($\alpha$), ($\beta$) the $\mathcal{P}'_{n,\ell} = \{a \cap \lambda : a \in \mathcal{P}_{n,\ell}\}$ are of the right form and so by (g)(γ) we finish.

Note that without loss of generality: if $a \in \mathcal{P}_{n,\ell}$ then $a \cap \operatorname{Reg} \cap (\lambda + 1) \setminus \mu \in \mathcal{P}_{n,\ell}$.

For $n = 0$ we can define N_0^a, N_0^b, $A_{n,\ell}$ trivially. Suppose N_m^a, N_m^b, $A_{m,\ell}$, $\mathcal{P}_{m,\ell}$ are defined for $m \leq n$, $\ell < \omega$ and f_m $(m < n)$ are defined. Now $\mathfrak{a}_n$ is well defined and $\subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu \subseteq \mathfrak{B}$ and $|\mathfrak{a}_n| \leq \theta$. So $\mathfrak{a}_n = \bigcup_\ell \mathfrak{a}_{n,\ell}$ and $\mathfrak{a}_{n,\ell} \subseteq \mathfrak{a}_{n,\ell+1}$ where $\mathfrak{a}_{n,\ell} =: \mathfrak{a}_n \cap A_{n,\ell}$ and, of course, $\mathfrak{a}_{n,\ell} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$ has cardinality $\leq \theta$. Note that $\mathfrak{a}_{n,\ell}$ is not necessarily in $\mathfrak{B}$ but

$(*)_1$ every countable subset of $\mathfrak{a}_{n,\ell}$ is included in some subset of $\mathfrak{B}$ which belongs to $\mathcal{P}_{n,\ell}$ and is $\subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$.

By the definition of $\lambda(3)$ (see "equivalently" there), for each n, ℓ we can find an increase sequence $\langle \mathfrak{a}_{n,\ell,k} : k < \omega \rangle$ of subsets of $\mathfrak{a}_{n,\ell}$ with union $\mathfrak{a}_{n,\ell}$ and $\mathfrak{d}_{n,\ell,k} \subseteq [\mu, \lambda(3)] \cap \mathrm{pcf}(\mathfrak{a}_{n,\ell,k})$, $|\mathfrak{d}_{n,\ell,k}| < \mu$ such that:

$(*)_2$ if $\mathfrak{b} \subseteq \mathfrak{a}_{n,\ell,k}$ is countable then $\mathfrak{b}$ is included in a finite union of some members of $\{\mathfrak{b}_\sigma[\mathfrak{a}_{n,\ell,k}] : \sigma \in \mathfrak{d}_{n,\ell,k}\}$ (hence $\max \mathrm{pcf}(\mathfrak{b}) \leq \lambda(3)$).

By the properties of pcf:

$(*)_3$ for each $\ell, k < \omega$ and $\mathfrak{c} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$ such that $\mathfrak{c} \in \mathcal{P}_{n,\ell}$ we can find $\mathfrak{e} = \mathfrak{e}_\mathfrak{c}^{\ell,k} \subseteq \lambda(3)^+ \cap \mathrm{pcf}\, \mathfrak{c}$, $|\mathfrak{e}| \leq |\mathfrak{d}_{n,\ell,k}| < \mu$ such that for every $\sigma \in \mathfrak{d}_{n,\ell,k}$ we have: $\mathfrak{c} \cap \mathfrak{b}_\sigma[\mathfrak{a}_{n,\ell,k}]$ is included in a finite union of members of $\{\mathfrak{b}_\tau[\mathfrak{c}] : \tau \in \mathfrak{e}_\mathfrak{c}\}$.

By [Sh371, 1.4] we can find $f_n \in \prod_{\sigma \in \mathfrak{a}_n} \sigma$ such that:

$(*)_4$ (α) $\sup(N_n^b \cap \sigma) < f_n(\sigma)$;

(β) if $\mathfrak{c} \in \mathcal{P}_{n,\ell}$, $\ell, k < \omega$, $\mathfrak{c} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$ and $\sigma \in \mathfrak{e}_\mathfrak{c}^{\ell,k} \subseteq \mathrm{pcf}(\mathfrak{c}) \cap [\mu, \lambda(3)]$ (where $\mathfrak{e}_\mathfrak{c}^{\ell,k}$ is from $(*)_3$) *then* for some $m < \omega$, $\sigma_p \in \sigma^+ \cap \mathrm{pcf}(\mathfrak{c})$ and $\alpha_p < \sigma_p$, (for $p \leq m$) the function $f_n \restriction (\mathfrak{b}_\sigma[\mathfrak{c}])$ is included in $\mathrm{Max}_{p \leq m} f_{\alpha_p}^{\mathfrak{c},\sigma_\ell} \restriction \mathfrak{b}_{\sigma_p}[\mathfrak{c}]$ (the Max taken pointwise).

Note

$(*)_5$ if $\mathfrak{b} \subseteq \mathfrak{a}_{n,\ell,k}$ is countable (where $\ell, k < \omega$) *then* there is $\mathfrak{c} \in \mathcal{P}_{n,\ell}$, $|\mathfrak{c}| < \mu$, $\mathfrak{c} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$ such that $\mathfrak{b} \subseteq \mathfrak{c}$.

By $(*)_4$:

$(*)_6$ if $\ell, k < \omega$, $\mathfrak{c} \in \mathcal{P}_{n,\ell}$, $\mathfrak{c} \subseteq \mathrm{Reg} \cap \lambda^+ \backslash \mu$, and $\sigma \in \mathfrak{d}_{n,\ell,k} \cap \lambda(3)^+ \cap \mathrm{pcf}\, \mathfrak{c} \backslash \mu$ then $f_n \restriction \mathfrak{b}_\sigma[\mathfrak{c}] \in \mathfrak{B}$.

You can check that (by $(*)_2 - (*)_6$) :

$(*)_7$ if $\mathfrak{b} \subseteq \mathfrak{a}_{n,\ell,k}$ is countable *then* there is $f_\mathfrak{b}^{n,\ell,k} \in \mathfrak{B}$, $|\mathrm{Dom}\ f_\mathfrak{b}^{n,\ell,k}| < \mu$ such that $f_n \restriction \mathfrak{b} \subseteq f_\mathfrak{b}^{n,\ell,k}$.

Let $\tau_i (i < \omega)$ list the Skolem function of $(H(\chi), \in, <_\chi^*)$. Let

$$A_{n+1,\ell} = \bigcup \{ \mathrm{Rang}\, (\tau_i \restriction (A_{n,j} \cup \mathrm{Rang}\ f_n \restriction \mathfrak{a}_{n,j,k})) : i < \ell, \quad j < \ell, \quad k < \ell \},$$

$$\mathcal{P}'_{n+1,\ell} = \bigcup_{m \leq \ell} \mathcal{P}_{n,m} \cup \{ f_n \restriction \mathfrak{a}' : \mathfrak{a}' \in \bigcup_{m \leq \ell} \mathcal{P}_{n,m} \text{ and } f_n \restriction \mathfrak{a}' \in \mathfrak{B} \},$$

and $\mathcal{P}_{n+1,\ell} = \mathcal{S}_{<\mu}(\lambda+1) \cap (\text{Skolem Hull of } A_{n+1,\ell} \cup \mathcal{P}'_{n+1,\ell} \cup (\theta+1))$.

So f_n, $\mathcal{P}_{n+1,\ell}$ are as required.

Thus we have carried the induction.

$\lambda(3) \leq \lambda(2)$: Let $\mathcal{P}$ exemplify the definition of $\lambda(2)$. Let $\mathfrak{a} \subseteq \text{Reg} \cap \lambda^+ \backslash \mu$, $|\mathfrak{a}| \leq \theta (< \mu)$. Let $J = J_{\leq \lambda(2)}[\mathfrak{a}]$, and let

$J_1 = \{\mathfrak{b}$: $\mathfrak{b} \subseteq \mathfrak{a}$ and there is $\langle \mathfrak{b}_i : i < i^* \rangle$, satisfying: $\mathfrak{b}_i \subseteq \mathfrak{b}$, $i^* < \mu$, $\max \text{pcf}\, \mathfrak{b}_i \leq \lambda(2)$ and any countable subset of $\mathfrak{b}$ is in the ideal which $\{\mathfrak{b}_i : i < i^*\}$ generates$\}$.

Clearly J_1 is an ideal of subsets of $\mathfrak{a}$ extending J. Let

$$J_2 = \left\{ \mathfrak{b}\text{: for some } \mathfrak{b}_n \in J_1 \text{ (for } n < \omega),\ \mathfrak{b} \subseteq \bigcup_n \mathfrak{b}_n \right\}.$$

Clearly J_2 is an $\aleph_1$-complete ideal extending J_1 (and J). If $\mathfrak{a} \in J_2$ we have that $\mathfrak{a}$ satisfies the requirement thus we have finished so we can assume $\mathfrak{a} \notin J_2$. As we can force by Levy $(\lambda(2)^+, 2^{\lambda(2)})$ (alternatively, replacing $\mathfrak{a}$ by [Sh355, §1]) without loss of generality $\lambda(2)^+ = \max \text{pcf}\, \mathfrak{a}$ and so $\text{tcf}(\prod \mathfrak{a}/J_2) = \text{tcf}(\prod \mathfrak{a}/J) = \lambda(2)^+$. Let $\bar{f} = \langle f_\alpha : \alpha < \lambda(2)^+ \rangle$ be $<_J$-increasing, $f_\alpha \in \prod \mathfrak{a}$, cofinal in $\prod \mathfrak{a}/J$. Let $\mathfrak{B} \prec (H(\chi), \in, <^*_\chi)$ be of cardinality $\lambda(2)$, $\lambda(2)+1 \subseteq \mathfrak{B}$, $\mathfrak{a} \in \mathfrak{B}$, $\bar{f} \in \mathfrak{B}$ and $\mathcal{P} \in \mathfrak{B}$. Let $\mathcal{P}' =: \mathfrak{B} \cap \mathcal{S}_{<\mu}(\lambda)$.

For $B \in \mathcal{P}'$ (so $|B| < \mu$) let $g_B \in \prod \mathfrak{a}$ be $g_B(\sigma) =: \sup(\sigma \cap B)$, so for some $\alpha_B < \lambda$, $g_B <_J f_{\alpha_B}$. Let $\alpha(*) = \sup\{\alpha_B : B \in \mathcal{P}\}$, clearly $\alpha(*) < \lambda(2)^+$. So $\bigwedge_{B \in \mathcal{P}} g_B <_J f_{\alpha(*)}$. Note: $\mathcal{P} \subseteq \mathcal{P}'$ (as $\mathcal{P} \in \mathfrak{B}$, $|\mathcal{P}| \leq \lambda(2)$, $\lambda(2)+1 \subseteq \mathfrak{B}$) and for each $B \in \mathcal{P}$, $\mathfrak{c}_B =: \{\sigma \in \mathfrak{a} : g_B(\sigma) \geq f_{\alpha(*)}(\sigma)\}$ is in J and $J \subseteq J_1 \subseteq J_2$. Apply the choice of $\mathcal{P}$ (i.e. it exemplifies $\lambda(2)$) to $A =: \text{Rang}\, f_{\alpha(*)}$, get $\langle A_n, \mathcal{P}_n : n < \omega \rangle$ as there. Let $\mathfrak{a}_n =: \{\sigma \in \mathfrak{a} : f_{\alpha(*)}(\sigma) \in A_n\}$, so $\mathfrak{a} = \bigcup_n \mathfrak{a}_n$, hence for some m, $\mathfrak{a}_m \notin J_2$ (as $\mathfrak{a} \notin J_2$, J_2 is $\aleph_1$-complete) hence $\mathfrak{a}_m \notin J_1$. As $\mathfrak{a} \in \mathfrak{B}$, $\mathcal{P} \in \mathfrak{B}$ clearly $\mathcal{P}_m \subseteq \mathfrak{B}$. So $\{\mathfrak{c}_B : B \in \mathcal{P}_m\}$ is a family of $< \mu$ subsets of $\mathfrak{a}$, each in J and every countable $\mathfrak{b} \subseteq \mathfrak{a}_m$ is included in at least one of them (as for some $B \in \mathcal{P}_m$, $\text{Rang}(f_{\alpha(*)} \restriction \mathfrak{b}) \subseteq B$, hence $\mathfrak{b} \subseteq \mathfrak{c}_B$). Easy contradiction.

$\lambda(3) \leq \lambda(0)$ IF $\text{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$: Let $\mathfrak{a} \subseteq \text{Reg} \cap \lambda^+ \backslash \mu$, $|\mathfrak{a}| \leq \kappa$, let $\langle \mathfrak{a}_\ell : \ell < \omega \rangle$ be as guaranteed by the definition of $\lambda(0)$, let $\mathcal{P}_\ell \subseteq \mathcal{S}_{<\aleph_1}(\mathfrak{a}_\ell)$ exemplify $\text{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$, for each $\mathfrak{b} \in \mathcal{P}_\ell$ we can find a finite $\mathfrak{e}_\mathfrak{b} \subseteq (\text{pcf}\, \mathfrak{a}_\ell) \cap \lambda^+ \backslash \mu$ such that $\mathfrak{b} \subseteq \bigcup\{\mathfrak{b}_\sigma[\mathfrak{a}_\ell] : \sigma \in \mathfrak{e}_\mathfrak{b}\}$ and $\{\mathfrak{b}_{\ell,i} : i < i^*\}$ enumerates $\{\mathfrak{e}_\mathfrak{b} : \mathfrak{b} \in \mathcal{P}_\ell\}$.

$\lambda(0) \leq \lambda(1)$: Similar to the proof of $\lambda(3) \leq \lambda(2)$. $\blacksquare_{1.2}$

1.3 CLAIM: *Assume* $\aleph_0 < \mathrm{cf}\,\lambda \le \theta < \lambda < \lambda^*$, $\mathrm{pp}(\lambda) \le \lambda^*$ *and*

$$\mathrm{cov}(\lambda^*, \lambda^+, \theta^+, 2) < \lambda^*.$$

Then $\mathrm{cov}(\lambda, \lambda, \theta^+, 2) < \lambda^*$.

Proof: Easy.

1.3A Definition: Assume $\lambda \ge \theta = \mathrm{cf}\,\theta > \kappa = \mathrm{cf}\,\kappa > \aleph_0$.

(1) $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$ if $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$ (see [Sh420, Def 2.1(1)]), and $\delta \in S(\bar{C}) \Rightarrow \delta = \sup(\mathrm{acc}\ C_\delta)$ (note: $\mathrm{acc}\,C_\delta \subseteq C_\delta$), and we do not allow (viii)$^-$ (in [Sh420, Definition 2.1(1)]), or replace it by:

(viii)* for some list $\langle a_i: i < \theta\rangle$ of $\bigcup_{\alpha \in S(\bar{C})} \mathcal{P}_\alpha$, we have: $\delta \in S(\bar{C})$, $\alpha \in \mathrm{acc}\ C_\delta$ implies $\{a \cap \beta: a \in \mathcal{P}_\delta, \beta \in a \cap \alpha\} \subseteq \{a_i: i < \alpha\}$.

(2) For $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$ we define a filter $\mathcal{D}^{\oplus}_{(\bar{C},\bar{\mathcal{P}})}(\lambda)$ on $[\mathcal{S}_{<\kappa}(\lambda)]^{<\kappa}$ (rather than on $\mathcal{S}_{<\kappa}(\lambda)$ as in [Sh420, 2.4]) (let $\chi = \beth_{\omega+1}(\lambda)$) :

$Y \in \mathcal{D}^{\oplus}_{(\bar{C},\bar{\mathcal{P}})}(\lambda)$ iff $Y \subseteq (\mathcal{S}_{<\kappa}(\lambda))^{<\kappa}$ and for some $x \in H(\chi)$ for every $\langle N_\alpha, N^*_a: \alpha < \theta,\ a \in \bigcup_{\delta \in S} \mathcal{P}_\delta\rangle$ satisfying condition $\otimes$ from [Sh420, 2.4], and also $[a \in \mathcal{P}_\delta \ \&\ \delta \in S \ \&\ \alpha < \theta \Rightarrow x \in N^*_a \ \&\ x \in N_\alpha]$ there is $A \in \mathrm{id}^a(\bar{C})$ such that $\delta \in S(\bar{C}) \backslash A \Rightarrow \langle \bigcup_{a \in \mathcal{P}_\delta} N^*_a \cap \lambda \cap N_\alpha : \alpha \in \mathrm{acc}\ C_\delta\rangle \in Y$.

Remark: For 1.3B below, see Definition of $\mathcal{T}^\ell(\theta, \kappa)$ and compare with [Sh420, Definition 2.1(2), (3)].

1.3B CLAIM:

(1) *If* $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$ *(so* $\lambda > \kappa$ *are regular uncountable) then* $D^{\oplus}_{(\bar{C},\bar{\mathcal{P}})}(\lambda)$ *is a non-trivial ideal on* $[\mathcal{S}_{<\kappa}(\lambda)]^{<\kappa}$.

(2) *If* $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$, $[\delta \in S(\bar{C}) \Rightarrow \delta = \sup \mathrm{acc}\ C_\delta]$, $\mathcal{P}_\delta = \{C_\delta \cap \alpha: \alpha \in C_\delta\}$ *then* $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$. *If* $\bar{C} \in \mathcal{T}^1[\theta, \kappa]$, $[\delta \in S(\bar{C}) \Rightarrow \delta = \sup \mathrm{acc}\ C_\delta]$ *and* $\mathcal{P}_\delta = \mathcal{S}_{<\aleph_0}(C_\delta)$ *then* $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$.

(3) *If* θ *is successor of regular,* $\sigma = \mathrm{cf}\,\sigma < \kappa$, *there is* $\bar{C} \in \mathcal{T}^0[\theta, \kappa] \cap \mathcal{T}^1[\theta, \kappa]$ *with: for* $\delta \in S(\bar{C})$, C_δ *is closed,* $\mathrm{cf}\,\delta = \sigma$ *and* $\mathrm{otp}\,C_\delta$ *divisible by* ω^2 *(hence* $\delta = \sup \mathrm{acc}\ C_\delta$*).*

(4) *Instead of "θ successor of regular", it suffices to demand*

(∗) $\quad \theta > \kappa$ *regular uncountable, and* $\displaystyle\bigwedge_{\alpha<\theta} \bigvee_{\kappa_1 \in [\kappa,\theta)} \mathrm{cov}(\alpha, \kappa_1, \kappa, 2) < \hat{\upsilon}$.

Replacing 2 by σ, "C_δ closed" is weakened to "$\{\mathrm{otp}(\alpha \cap C_\delta): \alpha \in C_\delta\}$ is stationary".

Proof: Check.

1.3C Claim: *Let $\lambda > \kappa = \mathrm{cf}\,\kappa > \aleph_0$, $\theta = \kappa^+$, $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$ then the following cardinals are equal:*

$$\mu(0) = \mathrm{cf}\,(\mathcal{S}_{<\kappa}(\lambda), \subseteq),$$
$$\mu(4) = \mathrm{Min}\left\{|Y|: Y \in \mathcal{D}^{\oplus}_{(\bar{C},\bar{\mathcal{P}})}(\lambda)\right\}.$$

Proof: Included in the proof of [Sh420, 2.6].

1.3D Claim: *Let $\lambda_1 \geq \lambda_0 > \kappa = \mathrm{cf}\,\kappa > \aleph_0$, $\theta = \kappa^+$ and $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^{\oplus}[\theta, \kappa]$. Let $\mathfrak{B}_{\lambda_1}$ be a rich enough model with universe λ_1 and countable vocabulary which is rich enough (e.g. all functions (from λ_1 to λ_1) definable in $(H(\beth_\omega(\lambda_1)^+), \in, <^*)$ with any finite number of places). Then the following cardinals are equal:*

$$\mu^*(0) = \mathrm{cov}(\lambda_1, \lambda_0^+, \kappa, 2),$$
$$\mu^+(4) = \mathrm{Min}\left\{|Y/\approx^{\lambda_0}_{\mathfrak{B}_{\lambda_1}}|: Y \in \mathcal{D}^{\oplus}_{(\bar{C},\bar{\mathcal{P}})}(\lambda_1)\right\} \text{ where } \langle a'_i\colon i \in \mathrm{acc}\ C_\delta\rangle \approx^{\lambda_0}_{\mathfrak{B}} \langle a''_i\colon i \in \mathrm{acc}\,C_\delta\rangle \text{ iff } \bigwedge_{i\in \mathrm{acc} C_\delta} \textit{Skolem Hull}_{\mathfrak{B}_{\lambda_1}}(a'_i \cup \lambda_0) = \textit{Skolem Hull}_{\mathfrak{B}_{\lambda_1}}(a'_i \cup \lambda_0).$$

Proof: Like the proof of [Sh420], 2.6, but using [Sh400, 3.3A].

2. Equality Relevant to Weak Diamond

It is well known that:

$$\kappa = \mathrm{cf}\,\kappa \,\&\, \theta > 2^{<\kappa} \Rightarrow \mathrm{cov}(\theta, \kappa, \kappa, 2) = \theta^{<\kappa} = \mathrm{cov}(\theta, \kappa, \kappa, 2)^{<\kappa}.$$

Now we have

2.1 Claim:

(1) *If $\mu > \lambda \geq \kappa$, $\theta = \mathrm{cov}(\mu, \lambda^+, \lambda^+, \kappa)$, $\mathrm{cov}(\lambda, \kappa, \kappa, 2) \leq \mu$ (or $\leq \theta$) then*

$$\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) = \mathrm{cov}(\theta, \kappa, \kappa, 2).$$

(2) *If in addition $\lambda \geq 2^{<\kappa}$ (or just $\theta \geq 2^{<\kappa}$) then*

$$\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2)^{<\kappa} = \mathrm{cov}(\mu, \lambda^+, \lambda^+, 2).$$

2.1A Remark:

(1) A most interesting case is $\kappa = \aleph_1$.
(2) This clarifies things in [Sh-f,AP1.17].

Proof: (1) Note that $\theta \geq \mu$ (because $\mu > \lambda \geq \kappa$). First we prove " $\leq$ ". Let $\mathcal{P}_0$ be a family of θ subsets of μ each of cardinality $\leq \lambda$, such that every subset of μ of cardinality $\leq \lambda$ is included in the union of $< \kappa$ of them (exists by the definition of $\theta = \mathrm{cov}(\mu, \lambda^+, \lambda^+, \kappa)$). Let $\mathcal{P}_0 = \{A_i: i < \theta\}$. Let $\mathcal{P}_1$ be a family of $\mathrm{cov}(\theta, \kappa, \kappa, 2)$ subsets of θ, each of cardinality $< \kappa$ such that any subset of θ of cardinality $< \kappa$ is included in one of them.

Let $\mathcal{P} =: \{\bigcup_{i \in a} A_i: a \in \mathcal{P}_1\}$; clearly $\mathcal{P}$ is a family of subsets of μ each of cardinality $\leq \lambda$, $|\mathcal{P}| \leq |\mathcal{P}_1| = \mathrm{cov}(\theta, \kappa, \kappa, 2)$, and every $A \subseteq \mu$, $|A| \leq \lambda$ is included in some union of $< \kappa$ members of $\mathcal{P}_0$ (by the choice of $\mathcal{P}_0$), say $\bigcup_{i \in b} A_i$, $b \subseteq \theta$, $|b| < \kappa$; by the choice of $\mathcal{P}_1$, for some $a \in \mathcal{P}_1$ we have $b \subseteq a$, hence $A \subseteq \bigcup_{i \in b} A_i \subseteq \bigcup_{i \in a} A_i \in \mathcal{P}$. So $\mathcal{P}$ exemplify $\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) \leq \mathrm{cov}(\theta, \kappa, \kappa, 2)$.

Second we prove the inequality $\geq$. If $\kappa \leq \aleph_0$ then $\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) = \theta$ and $\mathrm{cov}(\theta, \kappa, \kappa, 2) = \theta$ so $\geq$ trivially holds; so assume $\kappa > \aleph_0$. Obviously $\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) \geq \theta$. Note, if κ is singular then, as $\mathrm{cf}\,\lambda^+ > \lambda \geq \kappa$ for some $\kappa_1 < \kappa$, we have $\theta = \mathrm{cov}(\mu, \lambda^+, \lambda^+, \kappa) = \mathrm{cov}(\mu, \lambda^+, \lambda^+, \kappa')$ whenever $\kappa' \in [\kappa_1, \kappa]$ is a successor (by [Sh355, 5.2(8)]); also $\mathrm{cov}(\theta, \kappa, \kappa, 2) \leq \sup\{\mathrm{cov}(\theta, \kappa, \kappa', 2): \kappa' \in [\kappa_1, \kappa]$ is a successor cardinal$\}$ and $\mathrm{cov}(\theta, \kappa, \kappa', 2) \leq \mathrm{cov}(\theta, \kappa', \kappa', 2)$ when $\kappa' < \kappa$, so without loss of generality κ is regular uncountable. Hence for any $\theta_1 < \theta$ we have

$(*)_{\theta_1}$ we can find a family $\mathcal{P} = \{A_i: i < \theta_1\}$, $A_i \subseteq \mu$, $|A_i| \leq \lambda$, such that any subfamily of cardinality $\leq \lambda^+$ has a transversal. [Why? By [Sh355, 5.4], $(=^+)$ and [Sh355,1.5A] even for $\leq \mu$.]

Hence if $\theta_1 \leq \theta$, $\mathrm{cf}\,\theta_1 < \lambda^+$ (or even $\mathrm{cf}\,\theta_1 \leq \mu$) then $(*)_{\theta_1}$. Now we shall prove below

$$(\otimes_1) \qquad (*)_{\theta_1} \Rightarrow \mathrm{cov}(\theta_1, \kappa, \kappa, 2) \leq \mathrm{cov}(\mu, \lambda^+, \lambda^+, 2)$$

and obviously

$$(\otimes_2) \qquad \text{if } \mathrm{cf}\,\theta \geq \kappa \text{ then } \mathrm{cov}(\theta, \kappa, \kappa, 2) = \sum_{\alpha < \theta} \mathrm{cov}(\alpha, \kappa, \kappa, 2)$$

together; (as $\theta \leq \mathrm{cov}(\theta, \lambda^+, \lambda^+, 2)$ which holds as $\lambda < \mu \leq \theta$) we are done.

Proof of $\otimes_1$: Let $\{A_i: i < \theta_1\}$ exemplify $(*)_{\theta_1}$ and $\mathcal{P}_2$ exemplify the value of $\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2)$. Now for every $a \subseteq \theta_1$, $|a| < \kappa$, let $B_a =: \bigcup_{i \in a} A_i$; so $B_a \subseteq \mu$, $|B_a| \leq \lambda$ hence there is $A_a \in \mathcal{P}_2$ such that: $B_a \subseteq A_a$. Now for $A \in \mathcal{P}_2$ define $b[A] =: \{i < \theta_1: A_i \subseteq A\}$; it has cardinality $\leq \lambda$ (as any subfamily of $\{A_i: A_i \subseteq A\}$ of cardinality $\leq \lambda^+$ has a transversal). Note $a \subseteq b[A_a]$ (just read the definitions of $b[A]$ and A_a; note $a \in \mathcal{S}_{<\kappa}(\theta_1)$). For $A \in \mathcal{P}_2$ let $\mathcal{P}_A$ be a family of $\leq \mathrm{cov}(\lambda, \kappa, \kappa, 2)$ subsets of $b[A]$ each of cardinality $< \kappa$ such that any such set is included in one of them (exists as $|b[A]| \leq \lambda$ by the definition of $\mathrm{cov}(\lambda, \kappa, \kappa, 2)$). So for any $a \in \mathcal{S}_{<\kappa}(\theta_1)$ for some $c \in \mathcal{P}_{A_a}$, $a \subseteq c$. We can conclude that $\bigcup\{\mathcal{P}_A: A \in \mathcal{P}_2\}$ is a family exemplifying $\mathrm{cov}(\theta_1, \kappa, \kappa, 2) \leq \mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) + \mathrm{cov}(\lambda, \kappa, \kappa, 2)$ but the last term is $\leq \mu$ (by an assumption) whereas the first is $\geq \mu$ (as $\mu > \lambda$) hence the second term is redundant.

(2) By the first part it is enough to prove $\mathrm{cov}(\theta, \kappa, \kappa, 2)^{<\kappa} = \mathrm{cov}(\theta, \kappa, \kappa, 2)$, which is easy and well known (as $\theta \geq \mu > \lambda \geq 2^{<\kappa}$). $\blacksquare_{2.1}$

2.1B Remark: So actually if $\mu > \lambda \geq \kappa$, $\theta = \mathrm{cov}(\mu, \lambda^+, \lambda^+, \kappa)$ then ($\theta \geq \mu > \lambda \geq \kappa$ and)

$$\begin{aligned}\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) &\leq \mathrm{cov}(\mu, \lambda^+, \lambda^+, \kappa) + \mathrm{cov}(\theta, \kappa, \kappa, 2)\\ &= \theta + \mathrm{cov}(\theta, \kappa, \kappa, 2) = \mathrm{cov}(\theta, \kappa, \kappa, 2)\end{aligned}$$

and

$$\mathrm{cov}(\theta, \kappa, \kappa, 2) \leq \mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) + \mathrm{cov}(\lambda, \kappa, \kappa, 2),$$

hence, $\mathrm{cov}(\theta, \kappa, \kappa, 2) = \mathrm{cov}(\mu, \lambda^+, \lambda^+, 2) + \mathrm{cov}(\lambda, \kappa, \kappa, 2)$.

3. Cofinality of $S_{\leq\aleph_0}(\kappa)$ for κ Real Valued Measurable and Trees

In Rubin–Shelah [RuSh117] two covering properties were discussed concerning partition theorems on trees, the stronger one was sufficient, the weaker one necessary so it was asked whether they are equivalent. [Sh371, 6.1, 6.2] gave a partial positive answer (for λ successor of regular, but then it gives a stronger theorem); here we prove the equivalence.

In Gitik–Shelah [GiSh412] cardinal arithmetic, e.g. near a real valued measurable cardinal κ, was investigated, e.g. $\{2^\sigma: \sigma < \kappa\}$ is finite (and more); this section continues it. In particular we answer a problem of Fremlin: for κ real valued measurable, do we have $\mathrm{cf}(\mathcal{S}_{<\aleph_1}(\kappa), \subseteq) = \kappa$? Then we deal with trees with many branches; on earlier theorems see [Sh355, §0], and later [Sh410, 4.3].

3.1 Theorem: *Assume λ, θ, κ are regular cardinals and $\lambda > \theta = \kappa > \aleph_0$. Then the following conditions are equivalent:*

(A) *for every $\mu < \lambda$ we have $\mathrm{cov}(\mu, \theta, \kappa, 2) < \lambda$,*

(B) *if $\mu < \lambda$ and $a_\alpha \in \mathcal{S}_{<\kappa}(\mu)$ for $\alpha < \lambda$ then for some $W \subseteq \lambda$ of cardinality λ we have $|\bigcup_{\alpha \in W} a_\alpha| < \theta$.*

3.1A Remark: (1) Note that (B) is equivalent to: if $a_\alpha \in \mathcal{S}_{<\kappa}(\lambda)$ for $\alpha < \lambda$, then for some unbounded $S \subseteq \{\delta < \lambda\colon \mathrm{cf}(\delta) \geq \kappa\}$ and $b \in \mathcal{S}_{<\theta}(\lambda)$, for $\alpha \neq \beta$ in S, $a_\alpha \cap a_\beta \subseteq b$ (we can start with any stationary $S_0 \subseteq \{\delta < \lambda\colon \mathrm{cf}\,\delta \geq \kappa\}$, and use Fodour Lemma).

(2) We can replace everywhere θ by κ, but want to prepare for a possible generalization. By the proof we can strengthen "$W \subseteq \lambda$ of cardinality λ" to "$W \subseteq \lambda$ is stationary" (for $\neg$(A) $\rightarrow$ $\neg$(B) this is trivial, for (A) $\rightarrow$ (B) real), so these two versions of (B) are equivalent.

Proof:

(A)$\Rightarrow$(B):

Trivial [for $\mu < \lambda$ let $\mathcal{P}_\mu \subseteq \mathcal{S}_{<\theta}(\mu)$ exemplify $\mathrm{cov}(\mu, \theta, \kappa, 2) < \lambda$; suppose $\mu < \lambda$ and $a_\alpha \in \mathcal{S}_{<\kappa}(\mu)$ for $\alpha < \lambda$ are given, for each α for some $A_\alpha \in \mathcal{P}_\mu$ we have $a_\alpha \subseteq A_\alpha$; as $|\mathcal{P}_\mu| < \lambda = \mathrm{cf}\,\lambda$ for some A^* we have $W =: \{\alpha < \lambda\colon A_\alpha = A^*\}$ has cardinality λ, so S is as required in (B)].

$\neg$(A) $\Rightarrow$ $\neg$(B):

FIRST CASE: *For some $\mu \in [\theta, \lambda)$, $\mathrm{cf}\,\mu < \kappa < \mu$ and $pp^+_{<\kappa}(\mu) > \lambda$.* Then we can find $\mathfrak{a} \subseteq \mathrm{Reg} \cap \mu \backslash \theta$, $|\mathfrak{a}| < \kappa$, $\sup \mathfrak{a} = \mu$ and $\max \mathrm{pcf}_{J^{bd}_{\mathfrak{a}}} \mathfrak{a} \geq \lambda$. So by [Sh355, 2.3] without loss of generality $\lambda = \max \mathrm{pcf}\,\mathfrak{a}$; let $\langle f_\alpha\colon \alpha < \lambda\rangle$ be $<_{J_{<\lambda}[\mathfrak{a}]}$-increasing cofinal in $\prod \mathfrak{a}$.

Let $a_\alpha = \mathrm{Rang}(f_\alpha)$, so for $\alpha < \lambda$, a_α is a subset of $\mu < \lambda$ of cardinality $< \kappa$. Suppose $W \subseteq \lambda$ has cardinality λ, hence is unbounded, and we shall show that $\mu = |\bigcup_{\alpha \in W} a_\alpha|$; as $\mu \geq \theta$ this is enough. Clearly $a_\alpha = \mathrm{Rang}\ f_\alpha \subseteq \sup \mathfrak{a} = \mu$, hence $\bigcup_{\alpha \in W} a_\alpha \subseteq \mu$. If $|\bigcup_{\alpha \in W} a_\alpha| < \mu$ define $g \in \prod \mathfrak{a}$ by: $g(\sigma)$ is $\sup\left(\sigma \cap \bigcup_{\alpha \in W} a_\alpha\right)$ if $\sigma > |\bigcup_{\alpha \in W} a_\alpha|$ and 0 otherwise. So $g \in \prod \mathfrak{a}$ hence for some $\beta < \lambda$ $g < f_\beta \bmod J_{<\lambda}[\mathfrak{a}]$. As the f_β's are $<_{J_{<\lambda}[\mathfrak{a}]}$-increasing and $W \subseteq \lambda$ unbounded, without loss of generality $\beta \in W$, hence by g's choice $\left[\sigma \in \mathfrak{a} \backslash |\bigcup_{\alpha \in W} a_\beta|^+ \Rightarrow f_\beta(\sigma) \leq g(\sigma)\right]$ but $\left\{\sigma\colon \sigma \in \mathfrak{a}, \sigma > |\bigcup_{\theta \in W} a_\alpha|^+\right\} \notin J_{<\lambda}[\mathfrak{a}]$ (as μ is a limit cardinal and $\max \mathrm{pcf}_{J^{bd}_{\mathfrak{a}}}(\mathfrak{a}) \geq \lambda$), contradiction.

The main case is:

SECOND CASE: *For no $\mu \in [\theta, \lambda)$ is $\mathrm{cf}\,\mu < \kappa < \mu$, $pp^+_{<\kappa}(\mu) > \lambda$.* Let $\chi =: \beth_2(\lambda)^+$, $\mathfrak{B}$ be the model with universe λ and the relations and functions definable in $(H(\chi), \in, <^*_\chi)$ possibly with the parameters κ, θ, λ. We know that $\lambda > \theta^+$ (otherwise $\lambda = \theta^+$ and (A) holds). Let $S \subseteq \{\delta < \lambda\colon \mathrm{cf}\,\delta = \theta\}$ be stationary and in $I[\lambda]$ (see [Sh420, 1.5]) and let $S \subseteq S^+$, $\bar{C} = \langle C_\alpha : \alpha \in S^+ \rangle$ be such that: C_α closed, $\operatorname{otp} C_\alpha \le \theta$, $[\beta \in \operatorname{nacc} C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta]$, $[\operatorname{otp} C_\alpha = \kappa \Leftrightarrow \alpha \in S]$ and for $\alpha \in S^+$ limit, C_α is unbounded in α (see [Sh420, 1.2]).

Without loss of generality $\bar{C}$ is definable in $(\mathfrak{B}, \kappa, \theta, \lambda)$. Let $\mu_0 \in [\theta, \lambda)$ be minimal such that $\operatorname{cov}(\mu_0, \theta, \kappa, 2) \ge \lambda$, so $\mu_0 > \theta$, $\kappa > \operatorname{cf} \mu_0$. We choose by induction on $\alpha < \lambda$, $\mathfrak{A}_\alpha$, a_α such that:

(α) $\mathfrak{A}_\alpha \prec (H(\chi), \in, <^*_\chi)$, $\|\mathfrak{A}_\alpha\| < \lambda$ and $\mathfrak{A}_\alpha \cap \lambda$ is an ordinal and $\{\lambda, \mu_0, \theta, \kappa, \mathfrak{B}, \bar{C}\} \in \mathfrak{A}_\alpha$.

(β) $\mathfrak{A}_\alpha (\alpha < \lambda)$ is increasing continuous and $\langle \mathfrak{A}_\beta : \beta \le \alpha \rangle \in \mathfrak{A}_{\alpha+1}$.

(γ) $a_\alpha \in \mathcal{S}_{<\kappa}(\mu_0)$ is such that for no $A \in \mathcal{S}_{<\theta}(\mu_0) \cap \mathfrak{A}_\alpha$ is $a_\alpha \subseteq A$.

(δ) $\langle a_\beta : \beta \le \alpha \rangle \in \mathfrak{A}_{\alpha+1}$.

There is no problem to carry the definition and let $\mathfrak{A} = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$. Clearly it is enough to show that $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ contradict (B). Clearly $\mu_0 \in (\theta, \lambda)$ and $a_\alpha \in \mathcal{S}_{<\kappa}(\mu_0)$. So let $W \subseteq \lambda$, $|W| = \lambda$ and we shall prove that $|\bigcup_{\alpha \in W} a_\alpha| \ge \theta$. Note:

($*$) if $\mathfrak{a} \subseteq [\theta, \lambda)$, $|\mathfrak{a}| < \kappa$, $\mathfrak{a} \in \mathfrak{A}_\alpha$ (and $\mathfrak{a} \subseteq \mathrm{Reg}$, of course) then $(\prod \mathfrak{a}) \cap \mathfrak{A}_\alpha$ is cofinal in $\prod \mathfrak{a}$ (as $\max \operatorname{pcf} \mathfrak{a} < \lambda$).

Let $R = \{(\alpha, \beta) : \beta \in a_\alpha, \alpha < \lambda\}$ and

$$E =: \{\delta < \lambda : (\mathfrak{A}_\delta, R \restriction \delta, W \cap \delta, \mu_0) \prec (\mathfrak{A}, R, W, \mu_0) \quad \text{and} \quad \mathfrak{A}_\delta \cap \lambda = \delta\}.$$

Clearly E is a club of λ, hence we can find $\delta(*) \in S \cap \operatorname{acc}(E)$. Let $C_{\delta(*)} = \{\gamma_i : i < \theta\}$ (in increasing order). We now define by induction on $n < \omega$, M_n, $\langle N^n_\zeta : \zeta < \theta \rangle$, f_n such that:

(a) M_n is an elementary submodel of $(\mathfrak{A}, R, W)$, $\|M_n\| = \theta$,

(b) $\langle N^n_\zeta : \zeta < \theta \rangle$ is an increasing continuous sequence of elementary submodels of $\mathfrak{B}$,

(c) $\|N^n_\zeta\| < \theta$,

(d) $N^n_\zeta \in \mathfrak{A}_{\delta(*)}$,

(e) $\bigcup_{\zeta<\kappa} |N^n_\zeta| \subseteq |M_n|$,

(f) $f_n \in \prod(\mathrm{Reg} \cap M_n)$,

(g) $f_n(\sigma) > \sup(M_n \cap \sigma)$ for $\sigma \in \operatorname{Dom}(f_n) \backslash \theta^+$,

(h) for every $\zeta < \theta$, $f_n \restriction (\mathrm{Reg} \cap N^n_\zeta \backslash \theta^+) \in \mathfrak{A}_{\delta(*)}$,

(i) N^0_ζ is the Skolem Hull in $\mathfrak{B}$ of $\{\gamma_i, i : i < \zeta\}$,

(j) N^{n+1}_ζ is the Skolem Hull in $\mathfrak{B}$ of $N^n_\zeta \cup \{f_n(\sigma) : \sigma \in \mathrm{Reg} \cap N^n_\zeta \backslash \theta^+\}$,

(k) M_n is the Skolem Hull in $(\mathfrak{A}, R, W)$ of $\bigcup_{\ell<n} M_\ell \cup \bigcup_{\zeta<\theta} N^n_\zeta$.

There is no problem to carry the definition: for $n = 0$ define N^0_ζ by (i) [trivially (b) holds and also (c), as for (d), note that $\bar{C} \in \mathfrak{A}_0 \prec \mathfrak{A}_{\delta(*)}$ and $\{\gamma_i: i < \zeta\} \in \mathfrak{A}_{\delta(*)}$ as $\bar{C}$ is definable in $\mathfrak{B}$ hence $\{\langle \alpha, \gamma, \zeta\rangle: \alpha \in S^+, \zeta < \theta$, and γ is the ζ-th member of $C_\alpha\}$ is a relation of $\mathfrak{B}$ hence each $C_{\gamma_{\zeta+1}} (\zeta < \theta)$ is in $\mathfrak{A}_{\delta(*)}$ hence each $\{\gamma_i: i < \zeta\}$ is and we can compute the Skolem Hull in $\mathfrak{A}_{\gamma_j}$ for $j < \theta$ large enough].

Next, choose M_n by (k), it satisfies (e) + (a). If $\langle N^n_\zeta: \zeta < \theta\rangle$, M_n are defined, we can find f_n satisfying (f) + (g) + (h) by [Sh371,1.4] (remember (*)). For $n+1$ define N^n_ζ by (j) and then M_{n+1} by (k).

Next by [Sh400, 3.3A or 5.1A(1)] we have

$$(*) \quad \bigcup_{n<\omega} M_n \cap \delta(*) = \bigcup_{\substack{n<\omega\\ \zeta<\theta}} N^n_\zeta \cap \delta(*) \quad \text{hence} \quad \bigcup_{\substack{n<\omega\\ \zeta<\theta}} N^n_\zeta \cap W \text{ is unbounded in } \delta(*),$$

hence for some n

$$(*)_n \qquad \bigcup_{\zeta<\theta} N^n_\zeta \cap W \text{ is unbounded in } \delta(*).$$

Remember $N^n_\zeta \in \mathfrak{A}_{\delta(*)} = \bigcup_{\alpha<\delta(*)} \mathfrak{A}_\alpha = \bigcup_{i<\theta} \mathfrak{A}_{\gamma_i}$. So for some club e of θ we have:

$$(\otimes) \qquad \text{if } \zeta \in e, \quad \xi < \zeta \text{ then: } N^n_\xi \in \mathfrak{A}_{\gamma_\zeta}, \text{ and } \gamma_\zeta \in E \cap C_{\delta(*)}$$

(remember $\delta(*) \in \mathrm{acc}(E)$).

Hence, for $\zeta \in e$, we have: $\mathfrak{A}_{\gamma_\zeta} \cap \lambda = \gamma_\zeta$, and $W \cap N^n_\zeta \setminus \sup N^n_\xi \neq \emptyset$ for every $\xi < \zeta$. Let $e = \{\zeta(\epsilon): \epsilon < \theta\}$, $\zeta(\epsilon)$ strictly increasing continuous in ϵ. Now for every $\epsilon < \theta$, $N^n_{\zeta(\epsilon)} \in \mathfrak{A}_{\gamma_{\zeta(\epsilon+1)}}$ (and $\langle a_\beta: \beta \leq \sup(\lambda \cap N^n_{\zeta(\epsilon)}\rangle \in \mathfrak{A}_{\gamma_{\zeta(\epsilon+1)}}$) hence $A_1 =: \bigcup\{a_\beta: \beta \in W \cap N^n_{\zeta(\epsilon)}\} \subseteq A_2 =: \bigcup\{a_\beta: \beta \in N^n_{\zeta(\epsilon+1)}\} \cap \mu_0 \in \mathfrak{A}_{\gamma_{\zeta(\epsilon+1)}}$ and A_2 is a subset of μ_0 of cardinality $< \theta$ hence (by the choice of the a_γ's above) $a_{\gamma_{\zeta(\epsilon+1)}} \not\subseteq A_2$ hence $a_{\gamma_{\zeta(\epsilon+1)}} \not\subseteq \bigcup\{a_\beta: \beta \in W \cap N^n_{\zeta(\epsilon)}\}$; moreover, similarly $\gamma_{\zeta(\epsilon+1)} \leq \gamma < \lambda \Rightarrow a_\gamma \not\subseteq \bigcup\{a_\beta: \beta \in W \cap N^n_{\zeta(\epsilon)}\}$.

But $W \cap N^n_{\zeta(\epsilon+2)} \setminus \gamma_{\zeta(\epsilon+1)} \neq \emptyset$, hence $\langle \bigcup\{a_\beta: \beta \in W \cap N^n_{\zeta(\epsilon)}\}: \epsilon < \theta\rangle$ is not eventually constant, hence

$$\bigcup \left\{ a_\beta: \beta \in W \cap \bigcup_{\epsilon<\theta} N^n_{\zeta(\epsilon)} \right\} = \bigcup \left\{ a_\beta: \beta \in W \cap \bigcup_{\zeta<\theta} N^n_\zeta \right\}$$

has cardinality θ. Hence $\bigcup_{\beta\in W} a_\beta$ has cardinality $\geq \theta$, as required. $\blacksquare_{3.1}$

3.2 Conclusion: (1) If λ is real valued measurable then $\kappa = \mathrm{cf}\,[\mathcal{S}_{<\aleph_1}(\lambda), \subseteq]$ (equivalently, $\mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2) = \lambda$).
(2) Suppose λ is regular $> \kappa = \mathrm{cf}\,\kappa > \aleph_0$, I is a λ-complete ideal on λ extending J^{bd}_λ and is κ-saturated (i.e. we cannot partition λ to κ sets not in I). Then for $\alpha < \lambda$, $\mathrm{cf}(\mathcal{S}_{<\kappa}(\alpha), \subseteq) < \lambda$, equivalently $\mathrm{cov}(\alpha, \kappa, \kappa, 2) < \lambda$.

3.2A Remark: (1) So for regular $\theta \in (\kappa, \lambda)$ (in the above situation) we have $\bigwedge_{\alpha<\lambda} \mathrm{cov}(\alpha, \theta, \theta, 2) < \lambda$; actually $\kappa \leq \mathrm{cf}\,\theta \leq \theta < \lambda$ suffices by the proof.

Proof: (1) Follows by (2).
(2) The conclusion is (A) of Theorem 3.1, hence it suffices to prove (B). Let $\mu < \lambda$ and $a_\alpha \in \mathcal{S}_{<\kappa}(\mu)$ for $\alpha < \lambda$ be given. As $\kappa < \lambda = \mathrm{cf}\,\lambda$ without loss of generality for some $\sigma < \kappa$, $\bigwedge_{\alpha<\lambda} |a_\alpha| = \sigma$. Let f_α be a function from σ onto a_α, so $\mathrm{Rang}\, f_\alpha \subseteq \mu$. Now for each $i < \sigma$, $\langle \{\alpha < \lambda\colon f_\alpha(i) = \gamma\}\colon \gamma < \mu\rangle$ is a partition of λ to μ sets; as I is κ-saturated, $b_i =: \{\gamma < \mu\colon \{\alpha < \lambda\colon f_\alpha(i) = \gamma\} \notin I\}$ has cardinality $< \kappa$, hence $b =: \bigcup_{i<\sigma} b_i$ has cardinality $< \kappa + \sigma^+ \leq \kappa$ (remember $\sigma < \kappa = \mathrm{cf}\,\kappa$). For each $i < \sigma$, $\gamma \in \mu \backslash b_i$ the set $\{\alpha < \lambda\colon f_\alpha(i) = \gamma\}$ is in I; so as I is λ-complete, $\lambda > \mu$ we have: $\{\alpha < \lambda\colon f_\alpha(i) \notin b_i\}$ is in I. Now let

$$W =: \{\alpha < \lambda\colon \text{ for some } i < \sigma, f_\alpha(i) \notin b_i\} \subseteq \bigcup_{i<\sigma} \{\alpha < \lambda\colon f_\alpha(i) \notin b_i\}.$$

This is the union of $\leq \sigma < \lambda$ sets each in I, hence is in I, so $|\lambda \backslash W| = \lambda$, and clearly

$$\bigcup_{\alpha \in \lambda \backslash W} a_\alpha = \{f_\alpha(i)\colon \alpha \in \lambda \backslash W, i < \sigma\} \subseteq \{f_\alpha(i)\colon \alpha < \lambda, \neg f_\alpha(i) \notin b_i, i < \sigma\} \subseteq b,$$

and $|b| < \kappa$ so $\lambda \backslash W$ is as required in (B) of Theorem 3.1. $\blacksquare_{3.2}$

3.3 Lemma: *For every λ there is μ, $\lambda \leq \mu < 2^\lambda$ such that* (A) *or* (B) *or* (C) *below holds (letting $\kappa = \mathrm{Min}\{\theta\colon 2^\theta = 2^\lambda\}$) :*

- (A) *$\mu = \lambda$ and for every regular $\chi \leq 2^\lambda$ there is a tree T of cardinality $\leq \lambda$ with $\geq \chi$ $\mathrm{cf}(\kappa)$-branches (hence there is a linear order of cardinality $\geq \chi$ and density $\leq \lambda$).*
- (B) *$\mu > \lambda$ is singular, and:*
  - (α) $\mathrm{pp}(\mu) = 2^\lambda$ *(even $\lambda = \kappa \Rightarrow \mathrm{pp}^+(\mu) = (2^\lambda)^+$)*, $\mathrm{cf}\,\mu \leq \lambda$, $(\forall\theta)[\mathrm{cf}\,\theta \leq \lambda < \theta < \mu \Rightarrow \mathrm{pp}_\lambda\,\theta < \mu]$ *(and $\mu \leq 2^{<\kappa}$)*

hence

(α)′ *for every successor** $\chi \le 2^\lambda$ *there is a tree from [Sh355, 3.5]:* $\mathrm{cf}\,\mu$ *levels, every level of cardinality* $< \mu$ *and* χ $(\mathrm{cf}\,\mu)$*-branches,*

(β) *for every* $\chi \in (\lambda, \mu)$, *there is a tree* T *of cardinality* λ *with* $\ge \chi$ *branches of the same height,*

(γ) $\mathrm{cf}\,\mu \ge \mathrm{cf}\,\kappa$ *and even* $\mathrm{cf}\,\kappa > \aleph_0 \Rightarrow pp_{\Gamma(\mathrm{cf}\,\mu)}(\mu) =^+ 2^\lambda$.

(C) *Like (B) but we omit* (α) *and retain* (α)′.

Proof:

First Case: $\kappa = \aleph_0$. Trivially (A) holds.

Second Case: *κ is regular uncountable.* So $\kappa \le \lambda$ and $2^\kappa = 2^\lambda$ and $[\theta < \kappa \Rightarrow 2^\theta < 2^\kappa]$ hence $2^{<\kappa} < 2^\kappa$ (remember $\mathrm{cf}(2^\kappa) > \kappa$). Try to apply [Sh410, 4.3], its assumptions (i) + (ii) hold (with κ here standing for λ there) and if possibility (A) here fails then the assumption (iii) there holds, too; so there is μ as there; so (α), (γ) of (B) of 3.3 holds** and let us prove (β), so assume $\chi \in (\lambda, \mu)$, without loss of generality, is regular, and we shall prove the statement in (β) of 3.3(B). Without loss of generality χ is regular and $\mu' \in (\lambda, \chi) \,\&\, \mathrm{cf}\,\mu' \le \lambda \Rightarrow \mathrm{pp}_\lambda(\mu') < \chi$; i.e. χ is $(\lambda, \lambda^+, 2)$-inaccessible. [Why? If χ is not as required, we shall show how to replace χ by an appropriate regular $\chi' \in [\chi, \mu)$.]

Let $\mu' \in (\lambda, \chi)$ be minimal such that $\mathrm{pp}_\lambda(\mu') \ge \chi$, (so $\mathrm{cf}\,\mu' \le \lambda$) now $\mathrm{pp}(\mu') < \mu$ (by the choice of μ) and $\chi' =: \mathrm{pp}(\mu')^+$, by [Sh355, 2.3] is as required].

Let θ be minimal such that $2^\theta \ge \chi$. So trivially $\theta \le \kappa \le \lambda < \chi$ and $(2^{<\kappa})^\kappa = 2^\kappa$ hence $\mu \le 2^{<\kappa}$ hence $\chi < 2^{<\kappa}$; as χ is regular $< 2^{<\kappa}$ but $> \lambda \ge \kappa$, clearly $\theta < \kappa \le \lambda$; also trivially $2^{<\theta} \le \chi \le 2^\theta$ but χ is regular $> \lambda \ge \kappa > \theta$ and $[\sigma < \theta \Rightarrow 2^\sigma < \chi]$, so $2^{<\theta} < \chi \le 2^\theta$. Try to apply [Sh410, 4.3] with θ here standing for λ there; assumptions (i), (ii) there hold, and if assumption (iii) fails we get a tree with $\le \theta$ nodes *and* $\ge \chi$ θ-branches as required. So assume (iii) holds and we get there μ'; if $\mu' \le \lambda$ we have a tree as required; if

---

* If $\lambda = \kappa$, just regular, and we can change λ for this.

** Alternatively to quoting [Sh410, 4.3], we can get this directly, if $\mathrm{cov}(2^{<\kappa}, \lambda^+, (\mathrm{cf}\,\kappa)^+, \mathrm{cf}\,\kappa) < 2^\lambda$ we can get (A); otherwise by [Sh355, 5.4] for some $\mu_0 \in (\lambda, 2^{<\kappa}]$, $\mathrm{cf}(\mu_0) = \mathrm{cf}\,\kappa$ and $\mathrm{pp}(\mu_0) = (2^\lambda)$. Let $\mu \in (\lambda, 2^{<\kappa}]$ be minimal such that $\mathrm{cf}\,\mu \le \lambda \,\&\, \mathrm{pp}_\lambda(\mu) > 2^{<\kappa}$. Necessarily ([Sh355, 2.3] and [Sh371, 1.6(2), (3), (5)]) $\mathrm{pp}_\lambda(\mu) = \mathrm{pp}\,\mu = \mathrm{pp}(\mu_0) = (2^\lambda)$ and (again using [Sh355, 2.3]) we have $(\forall\theta)[\mathrm{cf}\,\theta \le \lambda < \theta < \mu \Rightarrow \mathrm{pp}_\lambda(\theta) < \mu]$; together ($\alpha$) of (B) holds. Also $\mu \le 2^{\le\kappa}$, hence $\mathrm{cf}(\mu) < \kappa \Rightarrow \mathrm{pp}\,\mu \le \mu^{<\kappa} \le 2^{<\kappa}$, contradiction, so ($\gamma$) of (B) follows from ($\alpha$). Note that if we replace λ by κ (changing the conclusion a little; or $\lambda = \kappa$) then by [Sh355, 5.4(2)] if 2^λ is regular the conclusion holds for $\chi = 2^\lambda$ too.

$\mu' \in (\lambda, 2^{<\theta}] \subseteq (\lambda, \chi)$ we get contradiction to "χ is $(\lambda, \lambda^+, 2)$-inaccessible" which, without loss of generality, we have assumed above.

THIRD CASE: *κ is singular (hence $2^{<\kappa}$ is singular,* $\mathrm{cf}(2^{<\kappa}) = \mathrm{cf}\,\kappa$). Let $\mu =: 2^{<\kappa}$ and we shall prove (C); easily (B)(γ) holds. Now ${}^{\kappa>}2$ is a tree with $2^{<\kappa} = \mu$ nodes and $2^\kappa = 2^\lambda$ κ-branches, so $(\alpha)'$ of (C) holds. As for (β) of (B), if κ is strong limit checking the conclusion is immediate, otherwise it follows from 3.4 part (3) below.

Clearly if $\mathrm{cf}\,\kappa > \aleph_0$, also (B) holds. $\blacksquare_{3.3}$

3.4 CLAIM:

(1) *Assume $\theta_{n+1} = \mathrm{Min}\{\theta: 2^\theta > 2^{\theta_n}\}$ for $n < \omega$ and $\sum_{n<\omega}\theta_n < 2^{\theta_0}$ (so θ_{n+1} is regular, $\theta_{n+1} > \theta_n$). Then: for infinitely many $n < \omega$, for some $\mu_n \in [\theta_n, \theta_{n+1})$ (so $2^{\mu_n} = 2^{\theta_n}$) we have:*

$(*)_{\mu_n,\theta_n}$ *for every regular $\chi \le 2^{\theta_n}$ there is a tree of cardinality μ_n with $\ge \chi$ θ_n-branches; if $\mu_n > \theta_n$ then $\mathrm{cf}(\mu_n) = \theta_n$, μ_n is $(\theta_n, \theta_n^+, 2)$-inaccessible.*

(2) *Moreover*

(α) *for every $n < \omega$ large enough for some μ_n :*

$$\theta_n \le \mu_n < \sum_{m<\omega}\theta_m \quad \text{and } (*)_{\mu_n,\theta_n} \quad \text{and } \mathrm{cf}(\mu_n) = \theta_n,$$

$$[\mu_n > \theta_n \Rightarrow \mu_n \text{ is } [(\theta_n, \theta_n^+, 2)\text{-inaccessible}, \mathrm{pp}(\mu_n) = 2^{\theta_n}].$$

(β) *Moreover, for infinitely many m we can demand: for every $n < m$, $\chi = \mathrm{cf}\,\chi \le 2^{\theta_n}$ the tree T^n_χ (witnessing $(*)_{\mu_n,\theta_n}$ for χ) has cardinality $< \theta_{m+1}$ (i.e. $\mu_m < \theta_{m+1}$).*

(3) *If κ is singular, $\kappa < 2^{<\kappa} < 2^\kappa$ then for every regular $\chi \in (\kappa, 2^{<\kappa})$, there is a tree with $< \kappa$ nodes and $\ge \chi$ branches (of same height). Also for some $\theta^* \in (\kappa, \mathrm{pp}^+(\kappa)) \cap \mathrm{Reg}$, for every regular $\chi \le 2^\kappa$ there is a tree T, $|T| \le \kappa^{\mathrm{cf}\,\kappa}$, with $\ge \chi$ θ^*-branches.*

Proof: Clearly (2) implies (1) and (3) (for (3) second sentence use ultraproduct). Let $\theta =: \sum_{n<\omega}\theta_n$. Let $S_0 =: \{n < \omega: (*)_{\theta_n,\theta_n} \text{ fails}\}$. Let for $n \in \omega \backslash S_0$, $\mu_n = \theta_n$ and note that (α) of 3.4(2) holds and if S_0 is co-infinite, also (β) of 3.4(2) holds. We can assume that S_0 is infinite (otherwise the conclusion of 3.4(2) holds). By [Sh355, 5.11], fully [Sh410, 4.3] for $n \in S_0$ there is μ_n such that:

$(\alpha)_n$ $\theta_n = \mathrm{cf}\,\mu_n < \mu_n \le 2^{<\theta_n}$,

$(\beta)_n$ $\mathrm{pp}_{\Gamma(\theta_n)}(\mu_n) \ge 2^{\theta_n}$ (hence equality holds and really $\mathrm{pp}^+_{\Gamma(\theta_n)}(\mu_n) = (2^{\theta_n})^+$) and

$(\gamma)_n$ $\theta_n < \mu' < \mu_n \& \mathrm{cf}\,\mu' \le \theta_n \Rightarrow \mathrm{pp}_{\le\theta_n}(\mu') < \mu_n$ hence $\mathrm{pp}^+_{\theta_n}(\mu_n) = \mathrm{pp}^+_{\Gamma(\theta_n)}(\mu_n)$
$= (2^{\theta_n})$.

Note that $2^{<\theta_n} = 2^{\theta_{n-1}}$ so $\mu_n \le 2^{\theta_{n-1}}$. By [Sh355, 5.11] for $n \in S_0$, part (α) (of 3.4(2)) holds except possibly $\mu_n < \theta$.

Remember $\mathrm{cf}(\mu_n) = \theta_n$.

Let $n < m$ be in S_0 and $\mu_n > \theta_m$, so $\mathrm{Max}\{\mathrm{cf}\,\mu_n, \mathrm{cf}\,\mu_m\} = \mathrm{Max}\{\theta_n, \theta_m\} < \mathrm{Min}\{\mu_n, \mu_m\}$ so by $(\gamma)_n$ (and [Sh355, 2.3(2)]) we have $\mu_n \ge \mu_m$. Note $\mathrm{cf}\,\mu_n = \theta_n$, $\mathrm{cf}\,\mu_m = \theta_m$ (which holds by $(\alpha)_n$, $(\alpha)_m$) hence $\mu_n > \mu_m$. As the class of cardinals is well ordered we get $S_1 =: \{n < \omega\colon n \in S_0, \mu_n \ge \theta_{n+1}\}$ is co-infinite and $S =: \{n\colon \mu_n \ge \theta\}$ is finite (so (α) of 3.4(2)(b) holds).

So for some $n(*) < \omega$, $S \subseteq n(*)$ hence for every $n \in [n(*), \omega)$ for some $m \in (n, \omega)$, $\mu_n < \theta_m$. Note: $n \ne m \Rightarrow \mu_n \ne \mu_m$ (as their cofinalities are distinct) and $[n \notin S_0 \Rightarrow \mu_n \notin \{\theta_m\colon m < \omega\}]$. Assume $n \ge n(*)$, if $\mu_n > \theta_{n+1}$, let $m = m_n = \mathrm{Min}\{m\colon \mu_{m+1} > \mu_n \text{ and } m \ge n\}$ (it is well defined as $\bigvee_k \mu_n < \theta_k$ and $\theta_k < \mu_k < \theta = \bigcup_{\ell<\omega} \theta_\ell$) and we shall show $\mu_m < \theta_{m+1}$; assume not, hence $m \in S_0$; so $\mu_{m+1} \le 2^{\theta_m} = \mathrm{pp}_{\Gamma(\theta_m)}(\mu_m) \le \mathrm{pp}_{\theta_{m+1}}(\mu_m)$ but $\mu_m \le \mu_n$ (by the choice of m) so as $\mathrm{cf}(\mu_m) = \theta_m \ne \theta_{m+1}$, necessarily $\mu_m > \theta_{m+1}$ and if $m+1 \notin S_0$ trivially and if $m+1 \in S_0$ by one of the demands on μ_{m+1} (in its choice) and [Sh355, 2.3] we have $\mu_{m+1} \le \mu_m$; but $\mu_m < \mu_n$, so $\mu_{m+1} < \mu_n$ contradicting the choice of m. So by the last sentence, $n \ge n(*) \Rightarrow \mu_{m_n} < \theta_{m_n+1}$. By [Sh355, 5.11] we get the desired conclusion (i.e. also part (β) of 3.4(2)). $\blacksquare_{3.4}$

Remark: It seemed that we cannot get more as we can get an appropriate product of a forcing notion as in Gitik and Shelah [GiSh344].

4. Bounds for $\mathrm{pp}_{\Gamma(\aleph_1)}$ for Limits of Inaccessibles*

4.1 Convention: For any cardinal μ, $\mu > \mathrm{cf}\,\mu = \aleph_1$ we let $\mathcal{Y}_\mu$, Eq_μ be as in [Sh420, 3.1], $\bar{\mu}$ is a strictly increasing continuous sequence of singular cardinals of cofinality $\aleph_0$ of length ω_1, $\mu = \sum_{i<\aleph_1} \mu_i$.

So μ stands here for μ^* in [Sh420, §3, §4, §5]. (Of course, $\aleph_1$ can be replaced by "regular uncountable".)

---

* In previous versions these sections have been in [Sh410], [Sh420] hence we use $\mathcal{Y}$, etc. (and not the context of [Sh386]); see 4.2B below.

4.2 THEOREM (Hypothesis [Sh420, 6.1C]*):

(1) *Assume*

(a) $\mu > \mathrm{cf}\,\mu = \aleph_1$, $\mathcal{Y} = \mathcal{Y}_\mu$, $Eq'_\mu \subseteq Eq_\mu$,

(b) *every* $D \in \mathrm{FIL}(\mathcal{Y})$ *is nice (see [Sh420, 3.5]),* $E = \mathrm{FIL}(\mathcal{Y})$ *(or at least there is a nice* $\mathcal{E}$ *(see [Sh420, 5.2–5],* $E = \bigcup \mathcal{E} = \mathrm{Min}\ \mathcal{E}$, $\mathcal{E}$ *is* μ*-divisible having weak* μ*-sums, but we concentrate on the first case),*

(c) $\mu < \lambda < \mathrm{pp}^+_E(\mu)$, λ *inaccessible.*

Then there are $e \in Eq_\mu$ *and* $\langle \lambda_x : x \in \mathcal{Y}/e \rangle$, *a sequence of inaccessibles* $< \mu$ *and a* $D \in \mathrm{FIL}(e, \mathcal{Y}) \cap E$ *nice to* μ, $D \in \mathrm{FIL}(e, \mathcal{Y}_\mu)$ *such that:*

(α) $\prod_{x \in \mathcal{Y}_\mu/e} \lambda_x / D$ *has true cofinality* λ,

(β) $\mu = \mathrm{tlim}_D \langle \lambda_x : x \in \mathcal{Y}_\mu \rangle$.

(2) *We can weaken "(b)" to "*$E \subseteq \mathrm{FIL}(Eq, \mathcal{Y})$ *and for* $D \in E$, *in the game* $wG(\mu, D, e, \mathcal{Y})$ *the second player wins choosing filters only from* E.

(3) *Moreover, for given* e_0, D_0, $\langle \lambda^0_x : x \in \mathcal{Y}/e_0 \rangle$, *if* $\prod_{x \in \mathcal{Y}/e_0} \lambda^0_x / D^e_0$ *is* λ*-directed, then without loss of generality* $e_0 \leq e$, $D_0 \leq D$ *and* $\lambda_x \leq \lambda_{x[e_0]}$.

4.2A Remark: (1) We could have separated the two roles of μ (in the definition of $\mathcal{Y}$, etc. and in $\lambda \in (\mu, \mathrm{pp}^+_E(\mu)))$ but the result is less useful; except for the unique possible cardinal appearing later.

(2) Compare with a conclusion of [Sh386] (see in particular 5.8 there):

THEOREM: *Suppose* $\lambda > 2^{\aleph_1}$, λ *(weakly) inaccessible.*

(1) *If* $\aleph_1 < \lambda_i = \mathrm{cf}\,\lambda_i < \lambda$ *for* $i < \omega_1$, D *is a normal filter on* ω_1, $\prod_{i<\omega_1} \lambda_i / D$ *is* λ*-directed, then for some* λ'_i, $\aleph_1 < \lambda'_i = \mathrm{cf}\,\lambda'_i \leq \lambda_i$ *and normal filter* D' *extending* D, $\lambda = \mathrm{tcf}\left(\prod_{i<\omega_1} \lambda'_i / D'\right)$ *and* $\{i : \lambda_i \text{ inaccessible}\} \in D'$.

(2) *If* $\aleph_1 = \mathrm{cf}\,\mu < \mu < \lambda$, $\mathrm{pp}_{\Gamma(\aleph_1)}(\mu) \geq \lambda$ *then for some* $\langle \lambda_i : i < \omega_1 \rangle$, $\aleph_1 < \lambda_i = \mathrm{cf}\,\lambda_i < \mu$, *each* λ_i *inaccessible and* $\lambda \in \mathrm{pcf}_{\Gamma(\aleph_1)}\{\lambda_i : i < \omega_1\}$.

Proof of 4.2: (1) By the definition of $\mathrm{pp}^+_E(\mu)$ (and assumption (c), and [Sh355, 2.3 (1) + (3)]) there are $D \in E$ and $f \in {}^{\mathcal{Y}_\mu/e}\mu$ such that:

$(\mathrm{A})_f$ $\mu > f(x) = \mathrm{cf}[f(x)] > \mu_{\iota(x)}$,

$(\mathrm{B})_{f,D}$ $\lambda = \mathrm{tcf}\left[\prod_{x \in \mathcal{Y}/e} f(x)/D\right]$.

Let $K_0 =: \{(f, D) : D \in E, f \in {}^{\mathcal{Y}_\mu/e}\mu$ and conditions $(\mathrm{A})_f$ and $(\mathrm{B})_{f,D}$ hold$\}$, so $K_0 \neq \emptyset$. Now if $(f, D) \in K_0$, for some γ

$(\mathrm{C})_{f,D,\gamma}$ in $G^\gamma(D, f, e, \mathcal{Y})$ the second player wins (see [Sh420, 3.4(2)])

---

* I.e.: if $\mathfrak{a} \subset \mathrm{Reg}$, $|\mathfrak{a}| < \min(\mathfrak{a})$, λ inaccessible then $\lambda > \sup(\lambda \cap \mathrm{pcf}\,\mathfrak{a})$.

hence $K_1 \neq \emptyset$ where $K_1 =: \{(f, D, \gamma) \in K_0$ condition $(\mathrm{C})_{f,D,\gamma}$ holds$\}$. Choose $(f^1, D_1, \gamma_{\langle\rangle}) \in K_1$ with $\gamma_{\langle\rangle}$ minimal. By the definition of the game

$(*)$ for every $A \neq \emptyset \bmod D_1$ we have $(f^1, D_1 + A, \gamma_{\langle\rangle}) \in K_1$.

Let $e_1 = e(D_1)$.

Case A: $\{x$: $f^1(x)$ *inaccessible*$\} \neq \emptyset \bmod D_1$. We can get the desired conclusion (by increasing D_1).

Case B: $\{x$: $f^1(x)$ *successor cardinal*$\} \neq \emptyset \bmod D_1$. By $(*)$, without loss of generality $f^1(x) = g(x)^+$, $g(x)$ a cardinal (so $\geq \mu_{\iota(x)}$) for every $x \in \mathcal{Y}_\mu/e$. By [Sh355, 1.3] for every regular $\kappa \in (\mu, \lambda)$ there is $f_\kappa \in {}^{(\mathcal{Y}/e)}\mathrm{Ord}$ satisfying:

(a) $f_\kappa < f^1$, each $f_\kappa(x)$ regular,

(b) $\mathrm{tlim}_{D_1} f_\kappa = \mu$,

(c) $\prod_x f_\kappa(x)/D_1$ has true cofinality κ.

By (a) we get

(d) $f_\kappa \leq g$.

By (b) we get, by the normality of D_1, that for the D_1-majority of $x \in \mathcal{Y}/e$, $f_\kappa(x) \geq \mu_{\iota(x)}$; as $f_\kappa(x)$ is regular (by (a)) and $\mu_{\iota(x)}$ singular (see 4.1) we get

(e) for the D_1-majority of $x \in \mathcal{Y}/e$, we have $f_\kappa(x) > \mu_{\iota(x)}$.

Let χ be large enough, let N be an elementary submodel of $(H(\chi), \in, <^*_\chi)$, $\lambda \in N$, $D_1 \in N$, $N \cap \lambda$ is the ordinal $\|N\|$ (singular for simplicity) and $\{\mu, \langle f^1, g, f_\kappa\colon \kappa \in \mathrm{Reg} \cap (\mu, \lambda)\rangle\}$ belongs to N. Choose $\kappa \in \mathrm{Reg} \cap \lambda \setminus (\sup \lambda \cap N)$, now in $\prod_{x \in \mathcal{Y}/e_1} f_\kappa(x)/D_1$, there is a cofinal sequence $\langle f_{\kappa,\zeta}\colon \zeta < \kappa\rangle$; as $\kappa > \sup(\lambda \cap N)$, so for some $\zeta(*) < \kappa$:

$$\otimes \quad h \in N \cap {}^{\mathcal{Y}/e_1}\mathrm{Ord} \Rightarrow \left\{x \in \mathcal{Y}/e_1\colon f_{\kappa,\zeta(*)}(x) \leq h(x) < f_\kappa(x)\right\} = \emptyset \bmod D_1.$$

[Why? For any such h define $h' \in {}^{\mathcal{Y}/e_1}\mathrm{Ord}$ by: $h'(x)$ is $h(x)$ if $h(x) < f_\kappa(x)$ and zero otherwise, so for some $\zeta_h < \kappa$, $h' < f_{\kappa,\zeta_h} \bmod D_1$. Let $\zeta(*) = \sup\left\{\zeta_h\colon h \in N \cap {}^{\mathcal{Y}/e_1}N\right\}$; it is $< \kappa$ as $\|N\| < \kappa$, and it is as required.]

Let $f_* = f_{\kappa,\zeta(*)}$. The continuation imitates [Sh371, §4], [Sh410, §5].

Let

$$\begin{aligned} K_2 = \Big\{(D, \bar{B}, \langle j_x\colon x \in \mathcal{Y}/e_1\rangle)\colon\ & D_1 \subseteq D \in E, \quad \text{player II wins } G_E^{\gamma_{\langle\rangle}}(f^1, D), \\ & e_1 = e(D), \bar{B} = \langle < B_{x,j}\colon j < j^0_x \leq \mu_{\iota(x)} > : x \in \mathcal{Y}/e_1\rangle \in N, \\ & |B_{x,j_x}| \lneq g(x) \ \text{ and } j_x < j^0_x \leq \mu_{\iota(x)}, \\ & \{x \in \mathcal{Y}/e_1\colon f_*(x) \ \text{ is in } B_{x,j_x}\} \in D\Big\}. \end{aligned}$$

Clearly $K_2 \neq \emptyset$. For each $(D, \bar{B}, \langle j_x : x \in \mathcal{Y}/e_1 \rangle) \in K_2$:

$(*)_1$ letting $h \in {}^{\mathcal{Y}/e_1}\mathrm{Ord}$, $h(x) = |B_{x,j_x}|$, for some $\bar{h} = \langle \langle \langle \rangle, f^1 \rangle, \langle \langle 0 \rangle, h \rangle \rangle$, for some $\gamma_{<0>} < \gamma_{<>}$ and D player II wins in $G_E^{\langle \gamma_{<>}, \gamma_{<0>} \rangle}(D, \bar{h}, e_1, \mathcal{Y}_\mu)$.

So choose $(D, \bar{B}, \langle j_x : x \in \mathcal{Y}/e_1 \rangle, \gamma_{\langle 0 \rangle})$ such that:

$(*)_2$ $(D, \bar{B}, \langle j_x : x \in \mathcal{Y}/e_1 \rangle) \in K_2$, $(*)_1$ for $\gamma_{\langle 0 \rangle}$ holds and (under those restrictions) $\gamma_{\langle 0 \rangle}$ is minimal.

So (as player I can "move twice"), for every $A \in D^+$, if we replace D by $D + A$, then $(*)_2$ still holds.

So without loss of generality (for the first and third members use normality):

$(*)_3$ one of the following sets belongs to D:

$$\begin{aligned} A_{0,\zeta} &= \left\{ x \in \mathcal{Y}/e_1 : \mathrm{cf}\, |B_{x,j_x}| > \mu_{\iota(x)} \text{ and } j_x^0 < \mu_\zeta \right\} \\ &\qquad \text{(for some } \zeta < \omega_1 \quad \text{such that } |\mathcal{Y}/e_1| < \mu_\zeta), \\ A_1 &= \left\{ x \in \mathcal{Y}/e_1 : \mathrm{cf}\, |B_{x,j_x}| < \mu_{\iota(x)} \leq |B_{x,j_x}| \right\}, \\ A_{2,\zeta} &= \{ x \in \mathcal{Y}/e_1 : |B_{x,j_x}| \leq \mu_\zeta \text{ and } j_x < \mu_\zeta \} \qquad \text{(for some } \zeta < \omega_1). \end{aligned}$$

If $A_{2,\zeta} \in D$ then (for $x \in \mathcal{Y}/e_1$)

$$B_x^* =: \bigcup \left\{ B_{x,j} : x \in \mathcal{Y}/e_1, j < j_x^0 \text{ and } |B_{x,j_x}| < \mu_\zeta \text{ and } j < \mu_\zeta \right\}$$

is a set of $\leq \mu_\zeta$ ordinals and

$$\{ x \in \mathcal{Y}/e_1 : f_*(x) \in B_x^* \} \in D$$

and $\langle B_x^* : x \in \mathcal{Y}/e_1 \rangle$ belongs to N (as $(D, \bar{B}, \langle j_x : x \in \mathcal{Y}/e_1 \rangle) \in K_2$ and the definition of K_2), contradiction to the choice of f_* (see $\otimes$, remember $D_1 \subseteq D$ by the definition of K_2).

If $A_1 \in D$, we can find $\bar{B}^1 \in N$, $\bar{B}^1 = \langle \langle B_{x,j}^1 : j < j_x^1 \leq \mu_{\iota(x)} \rangle : x \in \mathcal{Y}/e_1 \rangle$, $|B_{x,j}^1| \leq g(x)$ and $\bigwedge_{j < j_x^1} \left[\mathrm{cf}\, |B_{x,j}^1| \geq \mu_{\iota(x)} \vee |B_{x,j}^1| = 1 \right]$ and each $B_{x,j}$ satisfying $\mathrm{cf}\, |B_{x,j}| < \mu_{i(x)}$ is a union of $\mathrm{cf}\, |B_{x,j}|$ sets of the form B_{x,j^1}^1 of smaller cardinality and so for some $j_x^2 < j_x^1$, $f_*(x) \in B_{x,j_x} \Rightarrow f_*(x) \in B_{x,j_x^2}$ & $|B_{x,j_x^2}| < |B_{x,j_x}|$. Now playing one move in $G_E^{\langle \gamma_{<>}, \gamma_{<0>} \rangle}(D, \bar{h}, e, \mathcal{Y})$ we get contradiction to choice of $\gamma_{\langle 0 \rangle}$.

We are left with the case $A_{0,\zeta} \in D$, so without loss of generality $\bigwedge_{x,j} \mathrm{cf}\, |B_{x,j}| > \mu_{\iota(x)}$. Let

$$\mathfrak{a} = \left\{ \mathrm{cf}\, |B_{x,j}| : \mathrm{cf}\, |B_{x,j}| > \mu_{\iota(x)}, x \in \mathcal{Y}/e_1, j < j_x^0, j < \mu_\zeta \text{ and } \iota(x) > \zeta \right\},$$

so $\mathfrak{a}$ is a set of regular cardinals, and (remember $|\mathcal{Y}/e_1| < \mu_\zeta$) we have $|\mathfrak{a}| < \operatorname{Min} \mathfrak{a}$, so let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\theta[\mathfrak{a}]: \theta \in \operatorname{pcf} \mathfrak{a} \rangle$ be as in [Sh371, 2.6]. So as (by the Definition of K_2), $\langle \langle B_{x,j}: j < j^0_x \rangle: x \in \mathcal{Y}/e_1 \rangle \in N$, clearly $\mathfrak{a} \in N$ hence without loss of generality $\bar{\mathfrak{b}} \in N$. Let $\lambda^* = \sup[\lambda \cap \operatorname{pcf} \mathfrak{a}]$, so by Hypothesis $[420, 6.1(C)]$, $\lambda^* < \lambda$, but $\lambda^* \in N$, so $\lambda^* + 1 \subseteq N$.

By the minimality of the rank we have for every $\theta \in \lambda^* \cap \operatorname{pcf} \mathfrak{a}$, $\{x \in y/e_1: \operatorname{cf} |B_{x,j_x}| \in \mathfrak{b}_\theta\} = \emptyset \mod D$ hence $\prod_x \operatorname{cf} |B_{x,j_x}|/D$ is λ-directed, hence we get contradiction to the minimality of the rank of f_1.

(2), (3) Proof left to the reader. $\blacksquare_{4.2}$

4.2B Remark:

1. The proof of 4.3 below shows that in [Sh386] the assumption of the existence of nice filters is very weak, removing it will cost a little for at most one place.
2. We could have used the framework of [Sh386] but not for 4.3 (or use forcing).

4.3 CLAIM (Hypothesis 6.1(C) of [Sh420] even in any $K[A]$): *Assume $\mu > \operatorname{cf} \mu = \aleph_1$, $\mu > \theta > \aleph_1$, $\operatorname{pp}_{\Gamma(\theta,\aleph_1)}(\mu) \geq \lambda > \mu$, λ inaccessible. Then for some $e \in Eq_\mu$, $D \in \operatorname{FIL}(e, \mathcal{Y}_\mu)$ and sequence of inaccessibles $\langle \lambda_x: x \in \mathcal{Y}_\mu/e \rangle$, we have* $\operatorname{tlim}_D \lambda_x =$ *μ and $\lambda = \operatorname{tcf}(\prod \lambda_x/D)$ except perhaps for a unique λ in V (not depending on μ) and then $\operatorname{pp}^+_{\Gamma(\theta,\aleph_1)}(\mu) \leq \lambda^+$.*

Proof: By the Hyp. (see [Sh513, 6.12]) for some $\mathfrak{a} \subseteq \operatorname{Reg} \cap \mu$, $|\mathfrak{a}| < \operatorname{Min}(\mathfrak{a})$, $\lambda = \max \operatorname{pcf}(\mathfrak{a})$, and

$$(\forall \lambda' < \lambda)(\exists \mathfrak{b})[\mathfrak{b} \subseteq \mathfrak{a} \ \& \ |\mathfrak{b}| < \theta >] \& \lambda > \sup \operatorname{pcf}_{\aleph_1-\text{complete}}(\mathfrak{b}) > \lambda'],$$

$J = J_{<\lambda}[\mathfrak{a}]$. First assume "in $K[A]$ there is a Ramsey cardinal $> \lambda^\theta$ when $A \subseteq \lambda^\theta$". Choose $A \subseteq \lambda^\theta$ such that ${}^\theta\lambda \subseteq L[A]$ and for every $\alpha < \lambda^\theta$, there is a one to one function f_α from $|\alpha|$ (i.e. $|\alpha|^V$) onto α, $f_\alpha \in L[A]$, so $\operatorname{Card}^{L[A]} \cap (\lambda^\theta + 1) = \operatorname{Card}^V$, and apply 4.2 to the universe $K[A]$ (its assumption holds by [Sh420, 5.6]).

Second assume $(*)_\lambda$ "in $K[A]$ there is a Ramsey cardinal $> \lambda$ when $A \subseteq \lambda^+$" and assume our desired conclusion fails. Let $S \subseteq \lambda$ be stationary $[\delta \in S \Rightarrow \operatorname{cf} \delta = \theta^+]$, $\langle a_\alpha: \alpha < \lambda \rangle$, exemplify $S \in I[\lambda]$ (exist by [Sh420, §1]). We can find $\mathfrak{a}$, J as described above. Let $\langle f_\alpha: \alpha < \lambda \rangle$ exemplify $\lambda = \operatorname{tcf}(\prod \mathfrak{a}/J)$, now by [Sh355, 1.3] without loss of generality $\lambda = \max \operatorname{pcf} \mathfrak{a}$. Let $A_0 \subseteq \lambda$ be such that $\mathfrak{a}$, $\langle f_\alpha: \alpha < \lambda \rangle$, $\langle \mathfrak{b}_\sigma[\mathfrak{a}]: \sigma \in \operatorname{pcf} \mathfrak{a} \rangle$ are in $L[A_0]$. Hence in $L[A_0]$ for suitable J, $\langle f_\alpha/J: \alpha < \lambda \rangle$ is increasing, and without loss of generality for some $\langle \langle \mathfrak{c}^\delta_\alpha: \alpha \in a_\delta \rangle: \delta \in S \rangle \in L[A_0]$, we have: for $\delta \in S$, $\mathrm{cf}\,\delta = |\mathfrak{a}|^+$, a_δ a club of δ and $\langle f_\alpha \restriction (\mathfrak{a}\backslash \mathfrak{c}^\delta_\alpha)\colon \alpha \in a_\delta\rangle$ is $<$-increasing (see [Sh345b, 2.5] ("good point")) and $\mathfrak{c}^\delta_\alpha \in J$ and S is stationary in V, so the assumption of 4.3 holds in V^1 whenever $L[A_0] \subseteq V^1 \subseteq V$; hence for $A \subseteq \lambda^+$, in $K[A_0, A]$ the conclusion of 4.2 holds as we are assuming $(*)_\lambda$.

Note: if $A \subseteq \lambda$, in $K[A]$, $\lambda^{<\lambda} = \lambda$ hence if $\alpha < \lambda^+$, $A \subseteq \alpha$ then $K[A] \models$ "$\lambda^{<\lambda} < (\lambda^+)^V$".

Choose by induction on $\alpha < \lambda^+$ a set $A_\alpha \subseteq [\lambda\alpha, \lambda(\alpha+1))$ such that: A_0 is as above and for $\alpha > 0$: if $\langle \lambda_x\colon x \in \mathcal{Y}/e\rangle$, J exemplify the conclusion of 4.2 in $K\left[\bigcup_{\beta<\alpha} A_\beta\right]$, and $\langle f_i\colon i < \lambda\rangle$ exemplify the $\lambda = \mathrm{tcf}\left(\prod_{x\in\mathcal{Y}/e} \lambda_x/J\right)$, without loss of generality J canonical (all in $K\left[\bigcup_{\beta<\alpha} A_\beta\right]$, canonical means: the normal ideal generated by $\{x\colon \lambda_x \in \mathfrak{b}_{<\lambda}[\{\lambda_y\colon y \in \mathcal{Y}/e\}]\}$), then in $K\left[\bigcup_{\beta\le\alpha} A_\beta\right]$ we can find f, $\bigwedge_{\alpha<\lambda} f <_J \langle \lambda_x\colon x \in \mathcal{Y}/e\rangle$, $\bigwedge_\alpha f \not<_J f_\alpha$ (as they cannot exemplify the conclusion of 4.5 in V — otherwise we have finished).

Let $A = \bigcup_{\alpha<\lambda^+} A_\alpha$.

Now in $K[A]$ there are e, $\langle \lambda_x\colon \lambda \in \mathcal{Y}/e\rangle$, $\langle f_i\colon i < \lambda\rangle$ (and J) exemplifying the conclusion of 4.2 (by $(*)$ and [Sh513, 6.12(3)]). By 4.5 below, for some $\delta < \lambda^+$, e, $\langle \lambda_x\colon x \in \mathcal{Y}/e\rangle$, $\langle \mathfrak{b}_\sigma[\{\lambda_x\colon x \in \mathcal{Y}/e\}]\colon \sigma \in \mathrm{pcf}\{\lambda_x\colon x \in \mathcal{Y}/e\}\rangle$, $f_\alpha (\alpha < \lambda)$ all belongs to $K\left[\bigcup_{\gamma<\delta} A_\gamma\right]$, and in $K\left[\bigcup_{\gamma\le\delta} A_\gamma\right]$ we get a contradiction.

If $(*)_\lambda$ holds for every λ we are done. If not, let λ_0 be minimal such that $(*)_{\lambda_0}$ fails; so if $\lambda < \lambda_0$ the conclusion holds, and if $\lambda > \lambda_0$ then let $A \subseteq \lambda_0^+$ be such that in $K[A]$ there is no Ramsey, hence ([DoJ]) for $\mu \ge \lambda_0^+$ in V, $\mathrm{cov}(\mu, \theta, \theta, 2) \le \mu$, so the assumptions of 4.3 fail. Similarly $\mu > \theta$, $\mathrm{cf}(\mu) = \aleph_1$, $\mathrm{pp}_{\Gamma(\theta,\aleph_1)}(\mu) > \lambda_0^+$ bring a contradiction. $\blacksquare_{4.3}$

4.4 Conclusion: Hypothesis [Sh420, 6.1(C)] in any $K[A]$. (1) Assume $\mu > \mathrm{cf}\,\mu = \aleph_1$, $\mu_0 < \mu$, $\sigma \ge |\{\lambda\colon \mu_0 < \lambda < \mu,\ \lambda \text{ inaccessible}\}| < \mu$. Then

$$\sigma^{+4} > \left|\left\{\lambda\colon \mu < \lambda < \mathrm{pp}_{\Gamma(\sigma,\aleph_1)}(\mu) \text{ and } \lambda \text{ is inaccessible}\right\}\right|.$$

(2) The parallel of [Sh400, 4.3].

Proof: See [Sh410, 3.5] and use 4.2(3). $\blacksquare$

By [DoJe]

4.5 THEOREM: *If λ is regular ($> \aleph_1$) $A \subseteq \lambda$, $Z \in K[A]$ a bounded subset of λ then for some $\alpha < \lambda$, $Z \in \bigcup_{\alpha<\lambda} K[A \cap \alpha]$.*

We shall return to this elsewhere.

5. Densities of Box Products

5.1 Definition: $d_{<\kappa}(\lambda, \theta)$ is the density of the topological space ${}^{\lambda}\theta$ where the topology is generated by the following family of clopen sets:

$$\{[f]: f \in {}^{a}\theta \text{ for some } a \subseteq \lambda, |a| < \kappa\}$$

where

$$[f] = \{g \in {}^{\lambda}\theta: g \subseteq f\}.$$

So

$$d_{<\kappa}(\lambda, \theta) = \\ \operatorname{Min}\left\{|F|: F \subseteq {}^{\lambda}\theta \text{ and if } a \in \mathcal{S}_{<\kappa}(\lambda) \text{ and } g \in {}^{a}\theta \text{ then } (\exists f \in F) g \subseteq f\right\}.$$

If $\theta = 2$ we may omit it, if $\kappa = \aleph_0$ we may omit it (i.e. $d(\lambda, \theta) = d_{<\aleph_0}(\lambda, \theta)$). Always we assume $\lambda \geq \aleph_0$, $\kappa \geq \aleph_0, \theta > 1$ and $\lambda^+ \geq \kappa$. We write $d_{\kappa}(\lambda, \theta)$ for $d_{<\kappa^+}(\lambda, \theta)$.

5.1A Discussion: Note: for $\kappa = \aleph_0$ this is the Tichonov product, for higher κ those are called box products and d has obvious monotonicity properties.

$d\left(2^{\aleph_0}\right) = \aleph_0$ by the classical Hewitt–Marczewski–Pondiczery theorem [H], [Ma], [P]. This has been generalized by Engelking–Karlowicz [EK] and by Comfort–Negrepontis [CN1], [CN2] to show, for example, that $d_{<\kappa}(2^{\alpha}, \alpha) = \alpha$ if and only if $\alpha = \alpha^{<\kappa}$ ([CN1] (Theorem 3.1)). Cater–Erdős–Galvin [CEG] show that every non-degenerate space X satisfies $\mathrm{cf}(d_{<\kappa}(\lambda, X)) \geq \mathrm{cf}(\kappa)$ when $\kappa \leq \lambda^+$, and they note (in our notation) that "$d_{<\kappa}(\lambda)$ is usually (if not always) equal to the well-known upper bound $(\log \lambda)^{<\kappa}$". It is known (cf. [CEG], [CR]) that SCH $\Rightarrow d_{<\aleph_1}(\lambda) = (\log \lambda)^{\aleph_0}$, but it is not known whether $d_{<\aleph_1}(\lambda) = (\log \lambda)^{\aleph_0}$ is a theorem of ZFC.

The point in those theorems is the upper bound, as, of course, $d_{<\kappa}(\mu, \theta) > \chi$ if $\mu > 2^{\chi}$ & $\theta > 2$ [why? because if $F = \{f_i: i < \chi\}$ exemplify $d_{<\kappa}(\mu, \theta) \leq \chi$, the number of possible sequences $\langle \operatorname{Min}\{1, f_i(\zeta)\}: i < \chi \rangle$ (where $\zeta < \mu$) is $\leq 2^{\chi}$, so for some $\zeta \neq \xi$ they are equal and we get contradiction by g, $g(\zeta) = 0$, $g(\xi) = 1$, $\operatorname{Dom} g = \{\zeta, \xi\}$].

Also trivial is: for κ limit, $d_{<\kappa}(\lambda, \theta) = \kappa + \sup_{\sigma<\kappa} d_{<\sigma}(\lambda, \theta)$, so we only use κ regular; $d_{<\kappa}(\lambda, \theta) \geq \sigma^\theta$ for $\sigma < \kappa$.

Also if $\mathrm{cf}(\lambda) < \kappa$, λ strong limit then $d_{<\kappa}(\lambda) > \lambda$. The general case (say $2^{<\mu} < \lambda < 2^\mu$, $\mathrm{cf}\,\mu \leq \theta$) is similar; we ignore it in order to make the discussion simpler.

So the main problem is:

5.2 PROBLEM: *Assume λ is strong limit singular, $\lambda > \kappa > \mathrm{cf}(\lambda)$, what is $d_{<\kappa}(\lambda)$? Is it always 2^λ? Is it always $> \lambda^+$ when $2^\lambda > \lambda^+$?*

In [Sh93] this question was raised (later and independently) for model theoretic reasons. I thank Comfort for asking me about it in the Fall of '90.

5.3 LEMMA: *Suppose λ is singular strong limit, $\mathrm{cf}(\lambda) = \mathrm{cf}(\delta^*) \leq \delta^* < \mathrm{cf}(\kappa) \leq \kappa < \lambda$, $2 \leq \theta < \lambda$, $\lambda \leq \chi < 2^\lambda$ and $\langle \lambda_\alpha, \mu_\alpha, \chi_\alpha, \chi^*_\alpha : \alpha < \delta^* \rangle$ is such that:*

$\chi_\alpha = \theta^{\mu_\alpha}$, $\chi^*_\alpha = \mathrm{cov}(\chi_\alpha, \lambda_\alpha, \lambda_\alpha, 2)$,

$\alpha < \beta \Rightarrow \mu_\alpha < \mu_\beta$,

$\lambda = \bigcup_{\alpha<\delta^*} \mu_\alpha = \mathrm{tlim}_{\alpha<\delta} \lambda_\alpha$, $\theta < \mu_\alpha$,

$d_{<\kappa}(\mu_\alpha, \theta) \geq \lambda_\alpha$ *(this holds e.g. if* $(\forall \lambda' < \lambda_\alpha)[2^{\lambda'} < \mu_\alpha]$*)*,

$A_\alpha = [\mu_\alpha, \mu_\alpha + \mu_\alpha]$,

$G_\alpha = \{g$: *g a partial function from some* $a \in \mathcal{S}_{<\kappa}(A_\alpha)$ *to* $\theta\}$,

for $g \in G_\alpha$,

$[g] = \{f \in X_\alpha : g \subseteq f\}$ *where* $X_\alpha =: {}^{(A_\alpha)}\theta$, *so* $|X_\alpha| = \chi_\alpha$,

h_α *is a function from* $\mathcal{S}_{<\lambda_\alpha}({}^{(A_\alpha)}\theta)$ *to* G_α *such that* $h_\alpha(a)$ *"exemplifies" that a is not dense in* ${}^{(A_\alpha)}\theta$, *i.e.* $[f \in a \;\&\; g = h_\alpha(a) \Rightarrow g \not\subseteq f]$.

Then (F)$\Rightarrow$(E)$\Rightarrow$(D)$\Leftrightarrow$(C)$\Rightarrow$(B)$\Leftrightarrow$(A); and $(E)^\sigma$ decrease with σ and $(E)^\sigma \Rightarrow$(G) when $\chi^*_\alpha = \chi_\alpha$; and if every λ_α is regular (G)$\Rightarrow$(F) and if in addition $\bigwedge_{\alpha<\delta^*} \chi^*_\alpha = \chi_\alpha$ then (G)$\Leftrightarrow$(F)$\Leftrightarrow$(E), and if $\{\alpha < \delta^* : \sigma \leq \lambda_\alpha\} \neq \emptyset \bmod J$ and $\sigma < \lambda$ then (E)$\Leftrightarrow(E)^\sigma$ (fixing J), where

(A) $d_{<\kappa}(\lambda, \theta) > \chi$;

(B) *if $x_\zeta \in \prod_{\alpha<\delta^*} X_\alpha$ for $\zeta < \chi$ then there is $\bar{g} \in \prod_{\alpha<\delta^*} G_\alpha$ such that: for every $\zeta < \chi$, $\{\alpha < \delta^* : x_\zeta(\alpha) \notin [g_\zeta]\} \neq \emptyset$;*

(C) *if $x_\zeta \in \prod_{\alpha<\delta^*} X_\alpha$ for $\zeta < \chi$ then for some $w_\alpha \in \mathcal{S}_{<\lambda_\alpha}(X_\alpha)$ $(\alpha < \delta^*)$ for every $\zeta < \chi$, $\{\alpha < \delta^* : x_\zeta(\alpha) \in w_\alpha\} \neq \emptyset$;*

(D) *for every* $x_\zeta \in \prod_{\alpha<\delta^*} \chi_\alpha$ *for* $\zeta < \chi$ *there is* $\bar{w} \in \prod_{\alpha<\delta^*} \mathcal{S}_{<\lambda_\alpha}(\chi_\alpha)$ *such that: for each* $\zeta < \chi$, $\bigvee_{\alpha<\delta^*} x_\zeta(\alpha) \in w_\alpha$;

(E)$^\sigma$ *for some ideal* J *on* δ^* *extending* $J^{bd}_{\delta^*}$ *for every* $x_\zeta \in \prod_{\alpha<\delta^*} \chi_\alpha$ *(for* $\zeta < \chi$*) there are* $\epsilon(*) < \sigma$ *and* $\bar{w}^\epsilon \in \prod_{\alpha<\delta^*} \mathcal{S}_{<\lambda_\alpha}(\chi_\alpha)$ *for* $\epsilon < \epsilon(*)$ *such that for each* ζ *we have* $\bigvee_\epsilon \{\alpha < \delta^*: x_\zeta(\alpha) \notin w^\epsilon_\alpha\} = \emptyset$ *mod* J.
If $\sigma = 2$ *we may omit it;*

(F) *for some non-trivial ideal* J *on* δ^* *extending* $J^{bd}_{\delta^*}$ *we have*

$$\prod_{\alpha<\delta^*} \left(\mathcal{S}_{<\lambda_\alpha}(\chi_\alpha), \subseteq\right)/J \text{ is } \chi^+\text{-directed;}$$

(G) *for some non-trivial ideal* J *on* δ^* *extending* $J^{bd}_{\delta^*}$, *for any* $\langle \mathcal{P}_\alpha: \alpha < \delta^* \rangle$, $\mathcal{P}_\alpha$ *a* λ_α*-directed partial order of cardinality* $\leq \chi^*_\alpha$, *we have:* $\prod_{\alpha<\delta^*} \mathcal{P}_\alpha/J$ *is* χ^+*-directed.*

5.3A Remark:

(1) Note that the desired conclusion is 5.2(A).
(2) The interesting case of 5.3 is when $\{\mu_\alpha: \alpha < \delta^*\}$ does not contain a club of λ.
(3) Note that with notational changes we can arrange "λ is the disjoint union of $A_\alpha(\alpha < \delta^*)$, hence $\lambda_\theta = \prod_{\alpha<\delta^*} X_\alpha$".

Proof: Check. Clearly $(E)^\sigma$ decreases with σ, i.e. if $\sigma_1 < \sigma_2$ then $(E)^{\sigma_1} \Rightarrow (E)^{\sigma_2}$.

(E)$\Rightarrow$(D): Just for J varying on non-trivial ideals, we have monotonicity in J; and for $J = \{\emptyset\}$ we get (D).

(D)$\Leftrightarrow$(C): (C) is a translation of (D).

(C)$\Rightarrow$(B): If $x_\zeta \in \prod_{\alpha<\delta^*} X_\alpha$ for $\zeta < \chi$, let $\langle w_\alpha: \alpha < \delta^* \rangle$ be as in (C); for each α we know that w_α is not a dense subset of X_α (as $d_{<\kappa}(\mu_\alpha, \theta) \geq \lambda_\alpha > |w_\alpha|$) so there is $g_\alpha \in G_\alpha$ for which $[g_\alpha] \cap w_\alpha = \emptyset$, so $\bar{g} =: \langle g_\alpha: \alpha < \delta^* \rangle$ is as required in (B).

(B)$\Leftrightarrow$(A): They say the same (see 5.3A(3)).

(F)$\Rightarrow$(E): Note that (E) just says that in $\prod_{\alpha<\delta^*} (\mathcal{S}_{<\lambda_\alpha}(\chi_\alpha), \subseteq)$, any subset of $\{f: f \in \prod_{\alpha<\delta^*} \mathcal{S}_{<\lambda_\alpha}(\chi_\alpha)$, such that each $f(\alpha)$ is a singleton$\}$ has a $\leq_J$-upper bounded. In this form it is clearly a specific case of (F).

$(\mathrm{E})^\sigma \Rightarrow$(G) WHEN $\chi_\alpha = \chi^*_\alpha$: where $\{\alpha < \delta^*: \sigma \le \lambda_\alpha\} \ne \emptyset \bmod J$: Easy too.

Next assume every λ_α is regular, J an ideal on δ^*.

(G)$\Rightarrow$(F): (F) is a particular case of (G), because $(\mathcal{S}_{<\lambda_\alpha}(\chi_\alpha) \subseteq)$ is λ_α-directed as λ_α is regular and $\mathcal{S}_{<\lambda_\alpha}(\chi_\alpha)$ can be replaced by any cofinal subset and there is one of cardinality χ^*_α by its definition.
The rest should be clear. $\blacksquare_{5.3}$

5.4 CLAIM: *Assume λ is strong limit, $\theta < \lambda_0$, $\langle \lambda_\alpha: \alpha < \delta^* \rangle$, $\langle \chi^*_\alpha: \alpha < \delta^* \rangle$ are (strictly) increasing with limit λ, $\delta^* < \kappa \le \mathrm{cf}(\lambda) < \lambda$, $\lambda < \chi < 2^\lambda$ and $\lambda_\alpha \le \chi^*_\alpha$, λ_α regular for each $\alpha < \delta^*$. Then (G) of 5.3 holds (hence $d_{<\kappa}(\lambda,\theta) > \chi$) in any of the following cases:*

(a) *for some μ_α strong limit, $\mathrm{cf}(\mu_\alpha) < \kappa$, $2^{\mu_\alpha} = \mu^+_\alpha$, $\lambda_\alpha = \mu^+_\alpha$, $\chi^*_\alpha = \mu^+_\alpha$ and $\prod_{\alpha<\delta^*} \mu^+_\alpha / J$ is χ^+-directed,*

(b) *$k < \omega$ and for every α, $\chi^*_\alpha \le \lambda^{+k}_\alpha$ and for some ideal J on δ^*, for $\ell \le k$, $\prod \lambda^{+\ell}_\alpha / J$ is χ^+-directed, and $d_{<\kappa}(\chi^*_\alpha, \theta) \ge \lambda_\alpha$,*

(c) *for some $\gamma < \mathrm{cf}(\lambda)$ for every $\alpha < \delta^*$, $\chi^*_\alpha \le \lambda^{+\gamma}_\alpha$ and for some ideal J on δ^* for every $\zeta < \gamma$, $\prod_{\alpha<\delta^*}, \lambda^{+(\zeta+1)}_\alpha / J$ is χ^+-directed, and $d_{<\kappa}(\chi^*_\alpha, \theta) \ge \lambda_\alpha$,*

(d) *for some ideal J on δ^* extending $J^{bd}_{\delta^*}$ for every regular $\lambda'_\alpha \in [\lambda_\alpha, \chi^*_\alpha]$ satisfying $\mathrm{tlim}_J(\mathrm{cf}\,\lambda'_\alpha) = \lambda$, we have $\prod_{\alpha<\delta^*} \lambda'_\alpha / J$ is χ^+-directed and $d_{<\kappa}(\chi^*_\alpha, \theta) \ge \lambda_\alpha$.*

Proof: Clearly (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d).

Now the statements follow from the following observations 5.4A–5.7.

5.4A Observation: Assume that for $\alpha < \delta$, $\mathcal{P}_\alpha$ is a (non-empty) λ_α-directed partial order of cardinality χ_α, $|\delta|^+ < \lambda_\alpha = \mathrm{cf}(\lambda_\alpha) \le \chi_\alpha$, J an ideal on δ, $\theta^* = \mathrm{Min}\{\theta$: for some A and $\bar f$: $\bar f = \langle f_i: i < \theta \rangle$, $f_i \in \prod_{\alpha<\delta} \mathcal{P}_\alpha$ is $<_{J+A}$-increasing, $A \subseteq \delta$, $\delta \backslash A \notin J$ but for no $g \in \prod_{\alpha<\delta} \mathcal{P}_\alpha$, $\bigwedge_{i<\theta} \{\alpha: \mathcal{P}_\alpha \models f_i(\alpha) \le g(\alpha)\} \ne \emptyset \bmod (J+A)\}$. *Then* $\prod_{\alpha<\delta} \mathcal{P}_\alpha / J$ is θ^*-directed.

Proof: Without loss of generality no $\mathcal{P}_\alpha$ has a maximal element. If the conclusion of 5.4A fails, let F be a subset of $\prod_{\alpha<\delta} \mathcal{P}_\alpha$ with no $<_J$-upper bound, of minimal cardinality. Let $\theta = |F|$, so let $F = \{f_i: i < \theta\}$; by the choice of F without loss of generality $\alpha < \beta \Rightarrow f_\alpha <_J f_\beta$ hence θ is necessarily regular. If $\{\alpha < \delta: \lambda_\alpha \le \theta\} \in J$ we can find an upper bound: $g(\alpha)$ is a $\mathcal{P}_\alpha$-upper bound of $\{f_i(\alpha): i < \theta\}$ when $\lambda_\alpha > \theta$, and arbitrarily otherwise. So without loss of generality $\bigwedge_\alpha \lambda_\alpha \le \theta$. Now, remember $|\delta|^+ < \lambda_\alpha$, and so $|\delta|^+ < \theta$. By [Sh420, §1] we can find $\bar C = \langle C_i: i < \theta \rangle$,

$C_i \subset i$, $j \in C_i \Rightarrow C_j = j \cap C_i$, $\mathrm{otp}(C_i) \leq |\delta|^+$ and $S =: \{i < \lambda\colon \mathrm{cf}(i) = |\delta|^+, \delta = \sup(C_i)\}$ stationary: so wlog $j \in C_i \Rightarrow \bigwedge_{\alpha<\delta} \mathcal{P}_\alpha \models f_j(\alpha) < f_i(\alpha)$. Now we repeat the proof from [Sh282, 14]; better see [Sh345a, 2.6] or here 6.1.* $\blacksquare_{5.4A}$

5.5 Observation: In 5.4A, if A, $\bar{f}$ exemplify $\theta^* = \theta$ then

$$\theta^* \geq \min\{ \operatorname*{pre}_{J+A}(\bar{\chi}, \bar{\lambda})\colon A \subseteq \delta \text{ and } \delta \setminus A \notin J\}$$

where

5.6 Definition: For ideal I on δ and $\bar{\chi} = \langle \chi_\alpha\colon \alpha < \delta\rangle$, $\bar{\lambda} = \langle \lambda_\alpha\colon \alpha < \delta\rangle$, $\lambda_\alpha = \mathrm{cf}(\lambda_\alpha) \leq \chi_\alpha$ we let $\mathrm{pre}_I(\bar{\chi}, \bar{\lambda}) =: \mathrm{Min}\,\{|\mathcal{P}|\colon \mathcal{P}$ is a family of sequences of the form $\langle B_\alpha\colon \alpha < \delta\rangle$, $B_\alpha \subseteq \chi_\alpha$, $|B_\alpha| < \lambda_\alpha$ such that for every $g \in \prod_{\alpha<\delta} \chi_\alpha$ for some $\bar{B} \in \mathcal{P}$, $\{\alpha < \delta\colon g(\alpha) \in B_\alpha\} \neq \emptyset \bmod I\,\}$.

Proof: Check.

5.6A Remark: We use other parts of 5.3.

5.7 Observation: Let I be an ideal on δ^*, $\chi_\alpha \geq \lambda_\alpha > \delta^*$.

(1) Define $\mathcal{J}[I] = \{I + A\colon A \subseteq \delta, \delta\backslash A \notin I\}$.

(2) If $I_1 \subseteq I_2$, $\lambda^1_\alpha \geq \lambda^2_\alpha$, $\chi^1_\alpha \leq \chi^2_\alpha$ for $\alpha < \delta$ then $\mathrm{pre}_{I_1}(\bar{\chi}^1, \bar{\lambda}^1) \leq \mathrm{pre}_{I_2}(\bar{\chi}^2, \bar{\lambda}^2)$.

(3) If δ^* is the disjoint union of A_1, A_2, $A_\ell \notin I$ and $I_\ell =: I + A_\ell$ then $\mathrm{pre}_I(\bar{\chi}, \bar{\lambda}) = \mathrm{Min}\left\{\mathrm{pre}_{I_1}(\bar{\chi}, \bar{\lambda}), \mathrm{pre}_{I_2}(\bar{\chi}, \bar{\lambda})\right\}$.

(4) $\mathrm{pre}_I(\bar{\chi}^+, \bar{\lambda}) \leq \mathrm{pre}_I(\bar{\chi}, \bar{\lambda}) + \sup\{\mathrm{tcf}(\prod \chi^+_\alpha / I + A)\colon A \subseteq \delta, \delta\backslash A \notin I\}$.**
Moreover $\mathrm{pre}_I(\bar{\chi}^+, \bar{\lambda}) \leq \mathrm{Min}\{\mathrm{pre}_{I+A}(\bar{\chi}, \bar{\lambda}) + \mathrm{tcf}(\prod_{\alpha<\delta} \chi^+_\alpha / (I + A))\colon A \subseteq \delta, \delta \setminus A \notin I$ (and the tcf is well defined)$\}$.

(5) If each χ_α is a limit cardinal, $\mathrm{cf}\,\chi_\alpha > \delta^*$, then $\sup_{J\in\mathcal{J}[I]} \mathrm{pre}_J(\bar{\chi}, \bar{\lambda}) = \sup_{\bar{\chi}'<\bar{\chi}} \sup_{J\in\mathcal{J}[I]} \mathrm{pre}_J(\bar{\chi}', \bar{\lambda}) + \sup_{J\in\mathcal{J}[I]} \mathrm{tcf}(\Pi \chi_\alpha / I)$.

(6) $2^{|\delta^*|} + \sup_{J\in\mathcal{J}[I]} \sup\{\mathrm{tcf}(\Pi_{\alpha<\delta} \chi'_\alpha / J)\colon \lambda_\alpha \leq \chi'_\alpha = \mathrm{cf}(\chi'_\alpha) \leq \chi_\alpha$ and the true cofinality is well defined$\} \leq 2^{|\delta^*|} + \sup_{J\in\mathcal{J}[I]} \mathrm{pre}_J(\bar{\chi}, \bar{\lambda}) \leq 2^{|\delta^*|} + \sup_{J\in\mathcal{J}[I]} \sup\{\mathrm{tcf}(\Pi_{\alpha<\delta} \chi'_\alpha / J)\colon |\delta^*| < \mathrm{cf}(\chi'_\alpha)$ and $\lambda_\alpha \leq \chi'_\alpha \leq \chi_\alpha\}$.

(7) In part (6), if I is a precipitous ideal then the first inequality is equality.

Proof: Straightforward.

---

* In the main case here, $\bigwedge_\alpha 2^{|\delta^*|} < \lambda_\alpha$ and then trying all the possible A's, using their g's, the proof is very simple.

** Of course, $\bar{\chi}^+ = \langle \chi^+_\alpha\colon \alpha < \delta\rangle$.

5.9 Observation: In several of the models of set theory in which we know "λ strong, singular, limit, $2^\lambda > \lambda^{+}$" our sufficient conditions for $d_{\mathrm{cf}\,\lambda}(\lambda, 2) = 2^\lambda$ usually hold by the sufficient condition 5.4(a) (simplest: if GCH holds below λ, $\mathrm{cf}\,\lambda = \aleph_0$).

Remark: We could prove this consistency by looking more at the consistency proofs, adding many Cohen subsets to λ in preliminary forcing; but the present way looks more informative.**

6. Odds and Ends

6.1 LEMMA: *Suppose* $\mathrm{cf}(\delta) > \kappa^+$, *$I$ an ideal on κ, $f_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \delta$ is $\leq_I$-increasing. Then there are J_α, $\bar{s}$, $f'_\alpha(\alpha < \delta)$ such that:*

- (A) $\bar{s} = \langle s_i: i < \kappa\rangle$, *each s_i a set of $\leq \kappa$ ordinals,*
- (B) $\bigwedge_{i<\kappa} \bigwedge_{\alpha<\delta} \bigvee_{\beta\in s_i} f_\alpha(i) \leq \beta$,
- (C) $f'_\alpha \in \prod_{i<\kappa} s_i$ *is defined by* $f'_\alpha(i) = \mathrm{Min}[s_i \setminus f_\alpha(i)]$,
- (D) $\mathrm{cf}[f'_\alpha(i)] \leq \kappa$ *(e.g. $f'_\alpha(i)$ is a successor ordinal) implies* $f'_\alpha(i) = f_\alpha(i)$,

such that:

- (E) *J_α is an ideal on κ extending I (for $\alpha < \lambda$), decreasing with α (in fact for some $a_{\alpha,\beta} \subseteq \kappa$ (for $\alpha < \beta < \kappa$), $a_{\alpha,\beta}/I$ decreases with β, increases with α and J_α is the ideal generated by $I \cup \{a_{\alpha,\beta}: \alpha < \beta < \lambda\}$) so possibly $J_\alpha = \mathcal{P}(\kappa)$ and possibly $J_\alpha = I$,*
- (F) *if D is an ultrafilter on κ disjoint to J_α then f'_α/D is a $<_D$-l.u.b of $\langle f_\beta/D: \beta < \delta\rangle$ and $\{i < \kappa: \mathrm{cf}[f'_\alpha(i))] > \kappa\} \in D$,*
- (G) *if D is an ultrafilter on κ disjoint to I but for every α not disjoint to J_α then $\bar{s}$ exemplifies $\langle f_\alpha: \alpha < \delta\rangle$ is chaotic for D, i.e. for some club E of δ, $\beta < \gamma \in E \Rightarrow f_\beta \leq_D f'_\beta <_D f_\gamma$,*
- (H) *if $\mathrm{cf}(\delta) > 2^\kappa$ then $\langle f_\alpha: \alpha < \delta\rangle$ has a $\leq_I$-l.u.b. and even $\leq_I$-e.u.b,*
- (I) *if $b_\alpha =: \{i: f'_\alpha(i)$ has cofinality $\leq \kappa$ (e.g. is a successor)$\} \notin J_\alpha$ then: for every $\beta \in (\alpha, \delta)$ we have $f'_\alpha \restriction b_\alpha = f_\beta \restriction b_\alpha \bmod J_\alpha$.*

Moreover

$(\mathrm{F})^+$ *if $\kappa \notin J_\alpha$ then f'_α is an $<_{J_\alpha}$-e.u.b (= exact upper bound) of $\langle f_\beta: \beta < \delta\rangle$.*

Proof: Let $S = \{j: j \leq \sup \bigcup_{\alpha<\delta} \mathrm{Rang}(f_\alpha)$ has cofinality $\leq \kappa\}$, $\bar{e} = \langle e_j: j \in S\rangle$ be such that $[j = i+1 \Rightarrow e_j = \{i\}]$, $[j$ limit $\& j' \in S \cap e_j \Rightarrow e_{j'} \subseteq e_j]$, $e_j \subseteq j$ $[j$ limit $\Rightarrow j = \sup e_j]$ and $|e_j| \leq \kappa$.

** See much more on independence in a paper of Gitik and Shelah.

For a set $a \subseteq \sup \bigcup_{\alpha<\delta} \mathrm{Rang}\,(f_\alpha)$ let $\bar{e}[a] = a \cup \bigcup_{j \in a \cap S} e_j$ hence $\bar{e}[\bar{e}[a]] = \bar{e}[a]$ and $[a \subseteq b \Rightarrow \bar{e}[a] \subseteq \bar{e}[b]]$ and $|\bar{e}[a]| \le |a| + \kappa$. We try to choose by induction on $\zeta < \kappa^+$, the following: α_ζ, D_ζ, g_ζ, $\bar{s}_\zeta = \langle s_{\zeta,i}: i < \kappa \rangle$, $\langle f_{\zeta,\alpha}: \alpha < \delta \rangle$ such that:

(a) $g_\zeta \in {}^\kappa\mathrm{Ord}$,

(b) $s_{\zeta,i} = \bar{e}\,[\{g_\epsilon(i): \epsilon < \zeta\} \cup \{\sup_{\alpha<\delta} f_\alpha(i) + 1\}]$ so it is a set *of* $\le \kappa$ ordinals, increasing with ζ, $\sup_{\alpha<\delta} f_\alpha(i) + 1 \in s_{\zeta,i}$,

(c) $f_{\zeta,\alpha} \in {}^\kappa\mathrm{Ord}$, $f_{\zeta,\alpha}(i) = \mathrm{Min}[s_{\zeta,i} \backslash f_\alpha(i)]$,

(d) D_ζ is an ultrafilter on κ disjoint to I,

(e) for $\alpha < \delta$, $f_\alpha \le_{D_\zeta} g_\zeta$,

(f) α_ζ is an ordinal $< \delta$,

(g) $\alpha_\zeta \le \alpha < \lambda \Rightarrow g_\zeta <_{D_\zeta} f_{\zeta,\alpha}$.

If we succeed, let $\alpha(*) = \sup_{\zeta<\kappa^+} \alpha_\zeta$, so as $\mathrm{cf}(\delta) > \kappa^+$ clearly $\alpha(*) < \delta$. Now let $i < \kappa$ and look at $\langle f_{\zeta,\alpha(*)}(i): \zeta < \kappa^+ \rangle$; by its definition (see (c)), $f_{\zeta,\alpha(*)}(i)$ is the minimal member of the set $s_{\zeta,i} \backslash f_{\alpha(*)}(i)$. This set increases with ζ, so $f_{\zeta,\alpha(*)}(i)$ decreases with ζ (though not necessarily strictly), hence is eventually constant; so for some $\zeta_i < \kappa^+$ we have $\zeta \in [\zeta_i, \kappa^+) \Rightarrow f_{\zeta,\alpha(*)}(i) = f_{\zeta_i,\alpha(*)}(i)$. Let $\zeta(*) = \sup_{i<\kappa} \zeta_i$, so $\zeta(*) < \kappa^+$, hence

$$(*) \qquad \zeta \in [\zeta(*), \kappa^+) \Rightarrow \bigwedge_i f_{\zeta,\alpha(*)}(i) = f_{\zeta(*),\alpha(*)}(i) \Rightarrow f_{\zeta,\alpha(*)} = f_{\zeta(*),\alpha(*)}.$$

We know that $f_{\alpha(*)} \le_{D_{\zeta(*)}} g_{\zeta(*)} <_{D_{\zeta(*)}} f_{\zeta(*),\alpha(*)}$ hence for some i, $f_{\alpha(*)}(i) \le g_{\zeta(*)}(i) < f_{\zeta(*),\alpha(*)}(i)$, but $g_{\zeta(*)}(i) \in s_{\zeta(*)+1,i}$ hence $f_{\zeta(*)+1,\alpha(*)}(i) \le g_{\zeta(*)}(i) < f_{\zeta(*),\alpha(*)}(i)$, contradicting the choice of $\zeta(*)$.

So necessarily for some $\zeta < \kappa^+$ we are stuck, and clearly $s_{\zeta,i}(i < \kappa)$, $f_{\zeta,\alpha}(\alpha < \lambda)$ are well defined.

Let $s_i =: s_{\zeta,i}$ (for $i < \kappa$) and $f'_\alpha = f_{\zeta,\alpha}$ (for $\alpha < \lambda$). Clearly s_i is a set of $\le \kappa$ ordinals; now clearly:

$(*)_1$ $f_\alpha \le f'_\alpha$

$(*)_2$ $\alpha < \beta \Rightarrow f'_\alpha \le_I f'_\beta$,

$(*)_3$ if $b = \{i: f'_\alpha(\alpha) < f'_\beta(i)\} \notin I$, $\alpha < \beta < \delta$ then $f'_\alpha \restriction b <_I f_\beta \restriction b$.

We let for $\alpha < \delta$

$$J_\alpha = \Big\{ b \subseteq \kappa: b \in I \text{ or } b \notin I \quad \text{and for some } \beta \text{ we have: } \alpha < \beta < \delta \text{ and } f'_\alpha \restriction (\kappa \setminus b) =_I f'_\beta \restriction (\kappa \setminus b) \Big\}.$$

We let for $\alpha < \beta < \delta$, $a_{\alpha,\beta} =: \{i < \kappa: f'_\alpha(i) < f'_\beta(i)\}$. Then

$(*)_4$ J_α is an ideal on κ extending I, in fact is the ideal generated by $I \cup \{a_{\alpha,\beta}: \beta \in (\alpha,\delta)\}$.

As $\langle f'_\alpha: \alpha < \delta \rangle$ is $\leq_I$-increasing (i.e. $(*)_1$):

$(*)_5$ J_α decreases with α, in fact $a_{\alpha,\beta}/I$ increases with β, decreases with α,

$(*)_6$ if D is an ultrafilter on κ disjoint to J_α, then f'_α/D is a $<_D$-lub of $\{f_\beta/D: \beta < \delta\}$.

[Why? We know that $\beta \in (\alpha,\delta) \Rightarrow a_{\alpha,\beta} = \emptyset \bmod D$, so $f_\beta \leq f'_\beta =_D f'_\alpha$ for $\beta \in (\alpha,\delta)$, so f'_α/D is an $\leq_D$-upper bound. If it is not a least upper bound then for some $g \in {}^\kappa\mathrm{Ord}$, $\bigwedge_\beta f_\beta \leq_D g <_D f'_\alpha$ and we can get a contradiction to the choice of ζ, $\bar{s}$, f'_β as: (D,g) could serve as D_ζ, g_ζ.]

$(*)_7$ If D is an ultrafilter on κ disjoint to I but not to J_α (for every $\alpha < \lambda$) then $\bar{s}$ exemplifies $\langle f_\alpha: \alpha < \delta\rangle$ is chaotic for D.

[Why? For every $\alpha < \delta$ for some $\beta \in (\alpha,\delta)$ we have $a_{\alpha,\beta} \in D$, i.e. $\{i < \kappa: f'_\alpha(i) < f'_\beta(i)\} \in D$, so $\langle f'_\alpha/D: \alpha < \delta\rangle$ is not eventually constant, so if $\alpha < \beta$, $f'_\alpha <_D f'_\beta$ then $f'_\alpha <_D f_\beta$ (by $(*)_3$) and $f_\beta \leq_D f'_\beta$ (by (c)) as required.]

$(*)_8$ if $\kappa \notin J_\alpha$ then f'_α is an $\leq_{J_\alpha}$-e.u.b. of $\langle f_\beta: \beta < \delta\rangle$.

[Why? By $(*)_6$, f'_α is a $\leq_{J_\alpha}$-upper bound of $\langle f_\beta: \beta < \delta\rangle$; so assume that it is not a $\leq_{J_\alpha}$-e.u.b. of $\langle f_\beta: \beta < \delta\rangle$, hence there is a function g with domain κ, such that $g(i) < \mathrm{Max}\{1, f'_\alpha(i)\}$, but for no $\beta < \delta$ do we have

$$C_\beta =: \{i < \kappa: g(i) < \mathrm{Max}\{1, f_\beta(i)\} = \kappa \bmod J_\alpha.$$

Clearly $\langle C_\beta: \beta < \delta\rangle$ is increasing modulo J_α so there is an ultrafilter D on κ disjoint to $J_\alpha \cup \{C_\beta: \beta < \delta\}$. So $f_\beta \leq_D g \leq_D f'_\alpha$, so we get a contradiction to $(*)_6$ except when $g =_D f'_\alpha$ and then $f'_\alpha =_D O_\kappa$ (as $g(i) < 1 \vee g(i) < f'_\alpha(i)$). If we can demand $b^* = \{i: f'_\alpha(i) = 0\} \notin D$ we are done, but easily $b^* \setminus C_\beta \in J_\alpha$ so we finish.]

$(*)_9$ If $\mathrm{cf}[f'_\alpha(i)] \leq \kappa$ then $f'_\alpha(i) = f_\alpha(i)$.

[Why? By the definition of $s_\zeta = \bar{e}[\ldots]$ and the choice of $\bar{e}$, and $f'_\alpha(i)$.]

$(*)_{10}$ Clause (I) of the conclusion holds.

[Why? As $f_\alpha \leq_{J_\alpha} f_\beta \leq_{J_\alpha} f'_\alpha$ and $f_\alpha \restriction b =_{J_\alpha} f'_\alpha \restriction b$ by $(*)_9$.]

The reader can check the rest. $\blacksquare_{6.1}$

6.1A Example: We show that l.u.b and e.u.b are not the same. Let I be an ideal on κ, $\kappa^+ < \lambda = \mathrm{cf}(\lambda)$, $\bar{a} = \langle a_\alpha: \alpha < \lambda\rangle$ be a sequence of subsets of κ, (strictly) increasing modulo I, $\kappa \setminus a_\alpha \notin I$ but there is no $b \in \mathcal{P}(\kappa) \setminus I$ such that

$\bigwedge_\alpha b \cap a_\alpha \in I$. [Does this occur? E.g. for $I = \mathcal{S}_{<\aleph_0}(\omega)$, the existence of such $\bar{a}$ is known to be consistent; e.g. MA $\&\kappa = \aleph_0$ $\&$ $\lambda = 2^{\aleph_0}$. Moreover, for any κ and $\kappa^+ < \lambda = \mathrm{cf}\,\lambda \leq 2^\kappa$ we can find $a_\alpha \subseteq \kappa$ for $\alpha < \lambda$ such that, e.g., any Boolean combination of the a_α's has cardinality κ (less needed). Let I_0 be the ideal on κ generated by $\mathcal{S}_{<\kappa}(\kappa) \cup \{a_\alpha \backslash a_\beta : \alpha < \beta < \lambda\}$, and let I be maximal in $\{J: J$ an ideal on κ, $I_0 \subseteq J$ and $[\alpha < \beta < \lambda \Rightarrow a_\beta \backslash a_\alpha \notin J]\}$. So if G.C.H. fails, we have examples.] For $\alpha < \lambda$, we let $f_\alpha: \kappa \to \mathrm{Ord}$ be:

$$f_\alpha(i) = \begin{cases} \alpha & \text{if } \alpha \in \kappa \setminus a_i, \\ \lambda + \alpha & \text{if } \alpha \in a_i. \end{cases}$$

Now the constant function $f \in {}^\kappa\mathrm{Ord}$, $f(i) = \lambda + \lambda$ is a l.u.b of $\langle f_\alpha : \alpha < \lambda \rangle$ but not an e.u.b. (both $\bmod J$) (not e.u.b. is exemplified by $g \in {}^\kappa\mathrm{Ord}$ which is constantly λ).

6.2 CLAIM: *Suppose $\mu > \kappa = \mathrm{cf}\,\mu$, $\mu = \mathrm{tlim}_J\,\lambda_i$, $\delta < \mu$, $\lambda_i = \mathrm{cf}(\lambda_i) > \delta$ for $i < \delta$, J a σ-complete ideal on δ and $\lambda = \mathrm{tcf}\left(\prod_{i<\delta} \lambda_i / J\right)$, and $\langle f_\alpha : \alpha < \lambda \rangle$ exemplifies this.*

Then we have

- (*) *if $\langle u_\beta : \beta < \lambda \rangle$ is a sequence of pairwise disjoint non-empty subsets of λ, each of cardinality $\leq \sigma$ (not $< \sigma$!) and $\alpha^* < \mu$, then we can find $B \subseteq \lambda$ such that:*
  - (a) $\mathrm{otp}(B) = \alpha^*$,
  - (b) *if $\beta \in B$, $\gamma \in B$ and $\beta < \gamma$ then $\sup u_\beta < \min u_\gamma$,*
  - (c) *we can find $s_\zeta \in J$ for $\zeta \in \bigcup_{i \in B} u_i$ such that: if $\zeta \in \bigcup_{\beta \in B} u_\beta$, $\xi \in \bigcup_{\beta \in B} u_\beta$, $\zeta < \xi$ and $i \in \delta \backslash s_\zeta \backslash s_\xi$, then $f_\zeta(i) < f_\xi(i)$.*

Proof: For each regular $\theta, \theta^+ < \mu$, there is a stationary $S_\theta \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta < \delta\}$ which is in $I[\lambda]$ (see [Sh420, 1.5]) which is equivalent (see [Sh420, 1.2(1)]) to:

- (*) there is $\bar{C}^\theta = \langle C^\theta_\alpha : i < \lambda \rangle$,
  - (α) C^θ_α a subset of α, with no accumulation points (in C^θ_α),
  - (β) $[\alpha \in \mathrm{nacc}(C^\theta_\beta) \Rightarrow C^\theta_\alpha = C^\theta_\beta \cap \alpha]$,
  - (γ) for some club E^0_θ of λ,

$$[\delta \in S_\theta \cap E^0_\theta \Rightarrow \mathrm{cf}(\delta) = \theta < \delta \;\&\; \delta = \sup C^\theta_\delta \;\&\; \mathrm{otp}(C^\theta_\delta) = \theta].$$

Without loss of generality $S_\theta \subseteq E^0_\theta$, and $\bigwedge_{\alpha<\delta} \mathrm{otp}(C^\theta_\delta) \leq \theta$. By [Sh365, 2.3, Def. 1.3] for some club E_θ of λ, $\langle g\ell(C^\theta_\alpha, E_\theta) : \alpha \in S_\theta \rangle$ guess clubs (i.e. for every club $E \subseteq E_\theta$ of λ, for stationarily many $\zeta \in S_\theta$, $g\ell(C^\theta_\zeta, E_\theta) \subseteq E$) (remember $g\ell(C^\theta_\delta, E_\theta) = \{\sup(\gamma \cap E_\theta)\colon \gamma \in C^\theta_\delta; \gamma > \mathrm{Min}(E_\theta)\}$). Let $C^{\theta,*}_\alpha = \{\gamma \in C^\theta_\alpha\colon \gamma = \mathrm{Min}(C^\theta_\alpha \setminus \sup(\gamma \cap E_\theta))\}$, they have all the properties of the C^θ_α's and guess clubs in a weak sense: for every club E of λ for some $\alpha \in S_\theta \cap E$, if $\gamma_1 < \gamma_2$ are successive members of E then $|(\gamma_1, \gamma_2] \cap C^{\theta,*}_\alpha| \leq 1$; moreover, the function $\gamma \mapsto \sup(E \cap \gamma)$ is one to one on $C^{\theta,*}_\zeta$.

Now we define by induction on $\zeta < \lambda$, an ordinal α_ζ and functions $g^\zeta_\theta \in \prod_{i<\delta} \lambda_i$ (for each $\theta \in \Theta =: \{\theta\colon \theta < \mu,\ \theta \text{ regular uncountable}\}$).

For given ζ, let $\alpha_\zeta < \lambda$ be minimal such that:

$$\xi < \zeta \Rightarrow \alpha_\xi < \alpha_\zeta,$$
$$\xi < \zeta \;\&\; \theta \in \Theta \Rightarrow g^\zeta_\theta < f_{\alpha_\zeta} \bmod J.$$

Now α_ζ exists as $\langle f_\alpha\colon \alpha < \lambda\rangle$ is $<_J$-increasing cofinal in $\prod_{i<\lambda_i} / J$. Now for each $\theta \in \Theta$ we define g^ζ_θ as follows:

for $i < \delta^*$, $g^\zeta_\theta(i)$ is $\sup\left[\{g^\xi_\theta(i) + 1\colon \xi \in C^\theta_\zeta\} \cup \{f_{\alpha_\zeta}(i) + 1\}\right]$ if this number is $< \lambda_i$, and $f_{\alpha_\zeta}(i)$ otherwise.

Having made the definition we prove the assertion. We are given $\langle u_\beta\colon \beta < \lambda\rangle$, a sequence of pairwise disjoint non-empty subsets of λ, each of cardinality $< \sigma$ and $\alpha^* < \mu$. We should find B as promised; let $\theta =: (|\alpha^*| + |\delta|)^+$ so $\theta < \mu$ is regular $> |\delta|$. Let $E = \{\delta \in E_\theta : \text{for every } \zeta\colon [\zeta < \delta \Leftrightarrow \sup u_\zeta < \delta \Leftrightarrow u_\zeta \subseteq \delta \Leftrightarrow \alpha_\zeta < \delta]\}$. Choose $\alpha \in S_\theta \cap \mathrm{acc}(E)$ such that $g\ell(C^\theta_\zeta, E_\theta) \subseteq E$; hence letting $C^{\theta,*}_\alpha = \{\gamma_i\colon i < \theta\}$ (increasing) we know $\bigwedge_i (\gamma_i, \gamma_{i+1}) \cap E \neq \emptyset$. Now $B = \{\gamma_{5i+3}\colon i < \alpha^*\}$ are as required. For $\alpha \in \bigcup_{\zeta<\alpha^*} u_{5\zeta+3}$ let $s_\alpha = s^{\mathrm{o}}_\alpha \cup s^1_\alpha$. For $\alpha \in u_{5\zeta+3}$, $\zeta < \alpha^*$, let $s^{\mathrm{o}}_\alpha = \{i < \delta\colon g^{5\zeta+1}_\theta(i) < f_\alpha(i) < g^{5\zeta+4}(i)\}$, for each $\zeta < \alpha^*$; let $\langle \alpha_\epsilon\colon \epsilon < |u_{5\zeta+3}|\rangle$ enumerate $u_{5\zeta+3}$ and

$$s^1_{\alpha_\epsilon} = \{i\colon \text{ for every } \xi < \epsilon, f_{\alpha_\xi}(i) < f_{\alpha_\epsilon}(i) \Leftrightarrow \alpha_\xi < \alpha_\epsilon \Leftrightarrow f_{\alpha_\xi}(i) \leq f_{\alpha_\epsilon}(i)\}. \qquad \blacksquare_{6.2}$$

6.2A Remark: In 6.2: (1) We can avoid guessing clubs.

(2) Assume $\sigma < \theta_1 < \theta_2 < \mu$ are regular and there is $S \subseteq \{\delta < \lambda\colon \mathrm{cf}(\delta) = \theta_1\}$ from $I[\lambda]$ such that for every $\zeta < \lambda$ (or at least a club) of cofinality θ_2, $S \cap \zeta$ is stationary and $\langle f_\alpha\colon \alpha < \lambda\rangle$ obey suitable $\bar{C}^\theta$ (see [Sh345a, §2]). Then for some $A \subseteq \lambda$ unbounded, for every $\langle u_\beta\colon \beta < \theta_2\rangle$ sequence of pairwise disjoint non-empty subsets of A, each of cardinality $< \sigma$ with $[\min u_\beta, \sup u_\beta]$ pairwise disjoint we have: for every $B_0 \subseteq A$ of order type θ_2, for some $B \subseteq B_0$, $|B| = \theta_1$, (c) of $(*)$ of 6.2 holds.

(3) In $(*)$ of 6.2, "$\alpha^* < \mu$" can be replaced by "$\alpha^* < \mu^+$" (prove by induction on α^*).

6.3 OBSERVATION: *Assume $\lambda < \lambda^{<\lambda}$, $\mu = \mathrm{Min}\{\mu\colon 2^\mu > \lambda\}$. Then there are δ, χ and $\mathcal{T}$, satisfying the condition $(*)$ below for $\chi = 2^\mu$ or at least arbitrarily large regular $\chi \le 2^\mu$.*

$(*)$ *$\mathcal{T}$ a tree with δ levels, (where $\delta \le \mu$) with a set X of $\ge \chi$ δ-branches, and for $\alpha < \delta$, $\bigcup_{\beta<\alpha} |\mathcal{T}_\beta| < \lambda$.*

Proof of Observation: So let $\chi \le 2^\mu$ be regular, $\chi > \lambda$.

CASE 1: $\bigwedge_{\alpha<\mu} 2^{|\alpha|} < \lambda$. Then $\mathcal{T} = {}^{\mu>}2$, $\mathcal{T}_\alpha = {}^{\alpha}2$ are O.K. (the set of branches ${}^{\mu}2$ has cardinality 2^μ).

CASE 2: *Not Case 1.* So for some $\theta < \mu$, $2^\theta \ge \lambda$, but by the choice of μ, $2^\theta \le \lambda$, so $2^\theta = \lambda$, $\theta < \mu$ and so $\theta \le \alpha < \mu \Rightarrow 2^{|\alpha|} = 2^\theta$. Note $|{}^{\mu>}2| = \lambda$ as $\mu \le \lambda$.

SUBCASE 2A: $\mathrm{cf}(\lambda) \ne \mathrm{cf}(\mu)$. Let ${}^{\mu>}2 = \bigcup_{j<\lambda} B_j$, B_j increasing with j, $|B_j| < \lambda$. For each $\eta \in {}^{\mu}2$, (as $\mathrm{cf}(\lambda) \ne \mathrm{cf}(\mu)$) for some $j_\eta < \lambda$,

$$\mu = \sup\left\{\zeta < \mu\colon \eta \restriction \zeta \in B_{j_\eta}\right\}.$$

So as $\mathrm{cf}(\chi) > \mu$, for some ordinal $j^* < \lambda$ we have

$$\{\eta \in {}^{\mu}2\colon j_\eta \le j^*\} \text{ has cardinality } \ge \chi.$$

As $\mathrm{cf}(\lambda) \ne \mathrm{cf}(\mu)$ and $\mu \le \lambda$ (by its definition) clearly $\mu < \lambda$, hence $|B_{j^*}| \times \mu < \lambda$. Let

$$\mathcal{T} = \{\eta \restriction \epsilon\colon \epsilon < \ell g(\eta) \text{ and } \eta \in B_{j^*}\}.$$

It is as required.

SUBCASE 2B: *Not 2A so* $\mathrm{cf}(\lambda) = \mathrm{cf}(\mu)$. As $(\forall\sigma)[\theta \le \sigma < \mu \Rightarrow \lambda = 2^\sigma \Rightarrow \mathrm{cf}(\lambda) = \mathrm{cf}(2^\sigma) > \sigma]$, clearly $\mathrm{cf}(\lambda) \ge \mu$ so μ is regular. If $\lambda = \mu$ we get $\lambda = \lambda^{<\lambda}$ contradicting an assumption.

So $\lambda > \mu$, so λ singular. So if $\alpha < \mu$, $\mu < \sigma_i = \mathrm{cf}(\sigma_i) < \lambda$ for $i < \alpha$ then (see [Sh-g, 345a, 1.3(10)]) $\max\mathrm{pcf}\{\sigma_i\colon i < \alpha\} \le \prod_{i<\alpha} \sigma_i \le \lambda^{|\alpha|} \le \left(2^\theta\right)^{|\alpha|} \le 2^{<\mu} = \lambda$, but as λ is singular and $\max\mathrm{pcf}\{\sigma_i\colon i < \alpha\}$ is regular (see [Sh345a, 1.9]), clearly the inequality is strict, i.e. $\max\mathrm{pcf}\{\sigma_i\colon i < \alpha\} < \lambda$. So let $\langle \sigma_i\colon i < \mu\rangle$ be a strictly increasing sequence of regulars in (μ, λ) with limit λ, and by [Sh355, 3.4] there is $T \subseteq \prod_{i<\mu} \sigma_i$, $|\{\nu \restriction i\colon \nu \in T\}| \leq \max \mathrm{pcf}\{\lambda_j\colon j < i\} < \lambda$, and number of μ-branches $> \lambda$. In fact we can get any regular cardinal in $(\lambda, \mathrm{pp}^+(\lambda))$ in the same way. Let $\lambda^* = \min\{\lambda'\colon \mu < \lambda' \leq \lambda,\ \mathrm{cf}(\lambda') = \mu \text{ and } pp(\lambda') > \lambda\}$, so (by [Sh355, 2.3]), also λ^* has those properties and $\mathrm{pp}(\lambda^*) \geq \mathrm{pp}(\lambda)$. So if $\mathrm{pp}^+(\lambda^*) = (2^\mu)^+$ or $\mathrm{pp}(\lambda^*) = 2^\mu$ is singular, we are done. So assume this fails.

If $\mu > \aleph_0$, then (as in 3.4) $\alpha < 2^\mu \Rightarrow \mathrm{cov}(\alpha, \mu^+, \mu^+, \mu) < 2^\mu$ and we can finish as in subcase 2A (as in 3.4; actually $\mathrm{cov}(2^{<\mu}, \mu^+, \mu^+, \mu) < 2^\mu$ suffices which holds by the previous sentence and [Sh355, 5.4]). If $\mu = \aleph_0$ all is easy. $\blacksquare_{6.3}$

6.4 CLAIM: *Assume $\mathfrak{b}_k \subseteq \mathfrak{b}_{k+1} \subseteq \cdots$ for $k < \omega$, $\mathfrak{a} = \bigcup_{k\leq\omega} \mathfrak{b}_k$ (and $|\mathfrak{a}| < \mathrm{Min}\,\mathfrak{a}$) and $\lambda \in \mathrm{pcf}\,\mathfrak{a} \setminus \bigcup_{k<\omega} \mathrm{pcf}(\mathfrak{b}_k)$.*

1. *Then we can find finite $\mathfrak{d}_k \subseteq \mathrm{pcf}(\mathfrak{b}_k \setminus \mathfrak{b}_{k-1})$ (stipulating $\mathfrak{b}_{-1} = \emptyset$) such that $\lambda \in \mathrm{pcf} \bigcup_{k<\omega} \mathfrak{d}_k$.*
2. *Moreover, we can demand $\mathfrak{d}_k \subseteq (\mathrm{pcf}\,\mathfrak{b}_k) \setminus (\mathrm{pcf}(\mathfrak{b}_{k-1}))$.*

Proof: We start to repeat the proof of [Sh371, 1.5] for $\kappa = \omega$. But there we apply [Sh371, 1.4] to $\langle \mathfrak{b}_\zeta\colon \zeta < \kappa\rangle$ and get $\langle\langle \mathfrak{c}_{\zeta,\ell}\colon \ell \leq n_\zeta\rangle\colon \zeta < \kappa\rangle$ and let $\lambda_{\zeta,\ell} = \max \mathrm{pcf}(\mathfrak{c}_{\zeta,\ell})$. Here we apply the same claim ([Sh371, 1.4]) to $\langle \mathfrak{b}_k \setminus \mathfrak{b}_{k-1}\colon k < \omega\rangle$ to get part (1). As for part (2), in the proof of [Sh371, 1.5] we let $\delta = |\mathfrak{a}|^+ + \aleph_2$ choose $\langle N_i\colon i < \delta\rangle$, but now we have to adapt the proof of [Sh371, 1.4] (applied to $\mathfrak{a}$, $\langle \mathfrak{b}_k\colon k < \omega\rangle$, $\langle N_i\colon i < \delta\rangle$); we have gotten there, toward the end, $\alpha < \delta$ such that $E_\alpha \subseteq E$. Let $E_\alpha = \{i_k\colon k < \omega\}$, $i_k < i_{k+1}$. But now instead of applying [Sh371, 1.3] to each $\mathfrak{b}_\ell$ separately, we try to choose $\langle \mathfrak{c}_{\zeta,\ell}\colon \ell \leq n(\zeta)\rangle$ by induction on $\zeta < \omega$. For $\zeta = 0$ we apply [Sh371, 1.3]. For $\zeta > 0$, we apply [Sh371, 1.3] to $\mathfrak{b}_\zeta$ but there defining by induction on ℓ $\mathfrak{c}_\ell = \mathfrak{c}_{\zeta,\ell} \subseteq \mathfrak{a}$ such that $\max(\mathrm{pcf}(\mathfrak{a} \setminus \mathfrak{c}_{\zeta,0} \setminus \cdots \setminus \mathfrak{c}_{\zeta,\ell-1}) \cap \mathrm{pcf}\,\mathfrak{b}_\zeta)$ is strictly decreasing with ℓ. We use:

6.4A Observation: If $|\mathfrak{a}_i| < \mathrm{Min}(\mathfrak{a}_i)$ for $i < i^*$, then $\mathfrak{c} = \bigcap_{i<i^*} \mathrm{pcf}(\mathfrak{a}_i)$ has a last element or is empty.

Proof: Wlog $\langle |\mathfrak{a}_i|\colon i < i^* \langle$ is nondecreasing. By [Sh345b, 1.12]

$$(*)_1 \qquad \mathfrak{d} \subseteq \mathfrak{c}\ \&\ |\mathfrak{d}| < \mathrm{Min}\,\mathfrak{d} \Rightarrow \mathrm{pcf}(\mathfrak{d}) \subseteq \mathfrak{c}.$$

By [Sh371, 2.6]

$$\text{if } \lambda \in \mathrm{pcf}(\mathfrak{d}),\ \mathfrak{d} \subseteq \mathrm{pcf}(\mathfrak{c}),\ |\mathfrak{d}| < \mathrm{Min}(\mathfrak{d}) \text{ then}$$
$$\text{for some } \mathfrak{e} \subseteq \mathfrak{d} \text{ we have } |\mathfrak{e}| \leq \mathrm{Min}\,|\mathfrak{a}_0|,\ \lambda \in \mathrm{pcf}(\mathfrak{e}).$$

Now choose by induction on $\zeta < |\mathfrak{a}_0|^+$, $\theta_\zeta \in \mathfrak{c}$, satisfying $\theta_\zeta > \max\mathrm{pcf}\{\theta_\epsilon : \epsilon < \zeta\}$. If we are stuck in ζ, $\max\mathrm{pcf}\{\theta_\epsilon : \epsilon < \zeta\}$ is the desired maximum by $(*)_1$. If we succeed $\theta = \max\mathrm{pcf}\{\theta_\epsilon : \epsilon < |\mathfrak{a}_0|^+\}$ is in $\mathrm{pcf}\{\theta_\epsilon : \epsilon < \zeta\}$ for some $\zeta < |\mathfrak{a}_0|^+$ by $(*)_2$; easy contradiction. $\blacksquare_{6.4A}$

$\blacksquare_{6.4}$

6.5 Conclusion: Assume $\aleph_0 = \mathrm{cf}(\mu) \le \kappa \le \mu_0 < \mu$, $[\mu' \in (\mu_0, \mu) \,\&\, \mathrm{cf}(\mu') \le \kappa \Rightarrow \mathrm{pp}_\kappa(\mu') < \lambda]$ and $\mathrm{pp}^+_\kappa(\mu) > \lambda = \mathrm{cf}(\lambda) > \mu$. Then we can find λ_n for $n < \omega$, $\mu_0 < \lambda_n < \lambda_{n+1} < \mu$, $\mu = \bigcup_{n<\omega} \lambda_n$ and $\lambda = \mathrm{tcf}\prod_{n<\omega} \lambda_n / J$ for some ideal J on ω (extending J^{bd}_ω).

Proof: Let $\mathfrak{a} \subseteq (\mu, \mu) \cap \mathrm{Reg}$, $|\mathfrak{a}| \le \kappa$, $\lambda \in \mathrm{pcf}(\mathfrak{a})$. Without loss of generality $\lambda = \max\mathrm{pcf}\,\mathfrak{a}$, let $\mu = \bigcup_{n<\omega} \mu^0_n$, $\mu_0 \le \mu^0_n < \mu^0_{n+1} < \mu$, let $\mu^1_n = \mu^0_n + \sup\{\mathrm{pp}_\kappa(\mu') : \mu_0 < \mu' \le \mu^0_n \text{ and } \mathrm{cf}(\mu') \le \kappa\}$, by [Sh355, 2.3] $\mu^1_n < \mu$, $\mu^1_n = \mu^0_n + \sup\{\mathrm{pp}_\kappa(\mu') : \mu_0 < \mu' < \mu^1_n \text{ and } \mathrm{cf}(\mu') \le \kappa\}$ and obviously $\mu^1_n \le \mu^1_{n+1}$; by replacing by a subsequence without loss of generality $\mu^1_n < \mu^1_{n+1}$. Now let $\mathfrak{b}_n = \mathfrak{a} \cap \mu^1_n$ and apply the previous claim: to $\mathfrak{b}_k =: \mathfrak{a} \cap (\mu^1_n)^+$, note:

$$\max\mathrm{pcf}(\mathfrak{b}_k) \le \mu^1_k < \mathrm{Min}(\mathfrak{b}_{k+1} \setminus \mathfrak{b}_k). \qquad \blacksquare_{6.5}$$

6.6 CLAIM:

(1) *Assume* $\aleph_0 < \mathrm{cf}(\mu) = \kappa < \mu_0 < \mu$, $2^\kappa < \mu$ *and* $[\mu_0 \le \mu' < \mu \,\&\, \mathrm{cf}(\mu') \le \kappa \Rightarrow \mathrm{pp}_\kappa \mu' < \mu]$. *If* $\mu < \lambda = \mathrm{cf}(\lambda) < \mathrm{pp}^+(\mu)$ *then there is a tree* $\mathcal{T}$ *with* κ *levels, each level of cardinality* $< \mu$, $\mathcal{T}$ *has exactly* λ κ*-branches.*

(2) *Suppose* $\langle \lambda_i : i < \kappa \rangle$ *is a strictly increasing sequence of regular cardinals,* $2^\kappa < \lambda_0$, $\mathfrak{a} =: \{\lambda_i : i < \kappa\}$, $\lambda = \max\mathrm{pcf}\,\mathfrak{a}$, $\lambda_j > \max\mathrm{pcf}\{\lambda_i : i < j\}$ *for each* $j < \kappa$ *(or at least* $\sum_{i<\kappa} \lambda_i > \max\mathrm{pcf}\{\lambda_i : i < j\}$*) and* $\mathfrak{a} \notin J$ *where* $J = \{\mathfrak{b} \subseteq \mathfrak{a} : \mathfrak{b}$ *is the union of countably many members of* $J_{<\lambda}[\mathfrak{a}]\}$ *(so* $J \supseteq J^{bd}_{\mathfrak{a}}$, $\mathrm{cf}\,\kappa > \aleph_0$*). Then the conclusion of (1) holds with* $\mu = \sum_{i<\kappa} \lambda_i$.

Proof: (1) By (2) and [Sh371, §1] (or can use the conclusion of [Sh-g, AG 5.7]).

(2) For each $\mathfrak{b} \subseteq \mathfrak{a}$ define the function $g_{\mathfrak{b}} : \kappa \to \mathrm{Reg}$ by

$$g_{\mathfrak{b}}(i) = \max\mathrm{pcf}[\mathfrak{b} \cap \{\lambda_j : j < i\}].$$

Clearly $[\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \Rightarrow g_{\mathfrak{b}_1} \le g_{\mathfrak{b}_2}]$. As $\mathrm{cf}(\kappa) > \aleph_0$, J $\aleph_1$-complete, there is $\mathfrak{b} \subseteq \mathfrak{a}$, $\mathfrak{b} \notin J$ such that:

$$\mathfrak{c} \subseteq \mathfrak{b} \,\&\, \mathfrak{c} \notin J \Rightarrow \neg g_{\mathfrak{c}} <_J g_{\mathfrak{b}}.$$

Let $\lambda_i^* = \max \mathrm{pcf}(\mathfrak{b} \cap \{\lambda_j : j < i\})$. For each i let $\mathfrak{b}_i = \mathfrak{b} \cap \{\lambda_j : j < i\}$ and $\langle\langle f^{\mathfrak{b}}_{\lambda,\alpha} : \alpha < \lambda\rangle : \lambda \in \mathrm{pcf}\,\mathfrak{b}\rangle$ be as in [Sh371, §1]. Let

$$\mathcal{T}_i^0 = \left\{ \operatorname*{Max}_{\ell=1,n} f^{\mathfrak{b}}_{\lambda_\ell,\alpha_\ell} \restriction \mathfrak{b}_i : \lambda_\ell \in \mathrm{pcf}(\mathfrak{b}_i),\ \alpha_\ell < \lambda_\ell,\ n < \omega \right\}.$$

Let $\mathcal{T}_i = \{f \in \mathcal{T}_i^0$: for every $j < i, f \restriction \mathfrak{b}_j \in \mathcal{T}_j^0$ moreover for some $f' \in \prod_{j<\kappa} \lambda_j$, for every j, $f' \restriction j \in \mathcal{T}_i^0$ and $f \subseteq f'\}$, and $\mathcal{T} = \bigcup_{i<\kappa} \mathcal{T}_i$, clearly it is a tree, $\mathcal{T}_i$ its ith level (or empty), $|\mathcal{T}_i| \le \lambda_i^*$. By [Sh371, 1.3, 1.4] for every $g \in \prod \mathfrak{b}$ for some $f \in \prod \mathfrak{b}$, $\bigwedge_{i<\kappa} f \restriction \mathfrak{b}_i \in \mathcal{T}_i^0$ hence $\bigwedge_{i<\kappa} f \restriction \mathfrak{b}_i \in \mathcal{T}_i$. So $|\mathcal{T}_i| = \lambda_i^*$, and $\mathcal{T}$ has $\ge \lambda$ κ-branches. By the observation below we can finish (apply it essentially to $F = \{\eta$: for some $f \in \prod \mathfrak{b}$ for $i < \kappa$ we have $\eta(i) = f \restriction \mathfrak{b}_i$ and for every $i < \kappa$, $f \restriction \mathfrak{b}_i \in \mathcal{T}_i^0\}$), then find $A \subseteq \kappa$, $\kappa \setminus A \in J$ and $g^* \in \prod_{i<\kappa}(\lambda_i + 1)$ such that $Y' =: \{f \in F : f \restriction A < g^* \restriction A\}$ has cardinality λ and then the tree will be $\mathcal{T}'$ where $\mathcal{T}_i' =: \{f \restriction \mathfrak{b}_i : f \in Y'\}$ and $\mathcal{T}' = \bigcup_{i<\kappa} \mathcal{T}_i'$. (So actually this proves that if we have such a tree with $\ge \theta$ ($\mathrm{cf}(\theta) > 2^\kappa$) κ-branches then there is one with exactly θ κ-branches.)

6.6A Observation: (1) *If $F \subseteq \prod_{i<\kappa} \lambda_i$, J an $\aleph_1$-complete ideal on κ, and $[f \ne g \in F \Rightarrow f \ne_J g]$ and $|F| \ge \theta$, $\mathrm{cf}\,\theta > 2^\kappa$, then for some $g^* \in \prod_{i<\kappa}(\lambda_i + 1)$ we have:*

(a) *$Y = \{f \in F : f <_J g^*\}$ has cardinality θ,*

(b) *for $f' <_J g^*$, we have $|\{f \in F : f \le_J f'\}| < \theta$,*

(c) *there* are $f_\alpha \in Y$ for $\alpha < \theta$ such that: $f_\alpha <_J g^*$, $[\alpha < \beta < \theta \Rightarrow \neg f_\beta <_J f_\alpha]$.*

Proof: Let $Z =: \{g : g \in \prod_{i<\kappa}(\lambda_i + 1)$ and $Y_g =: \{f \in F : f \le_J g\}$ has cardinality $\ge \theta\}$. Clearly $\langle \lambda_i : i < \kappa\rangle \in Z$ so there is $g^* \in Z$ such that: $[g' \in Z \Rightarrow \neg g' <_J g^*]$; so (b) holds. Let $Y = \{f \in F : f <_J g^*\}$, easily $Y \subseteq Y_{g^*}$ and $|Y_{g^*} \setminus Y| \le 2^\kappa$ hence $|Y| \ge \theta$, also clearly $[f_1 \ne f_2 \in F\ \&\ f_1 \le_J f_2 \Rightarrow f_1 <_J f_2]$; if (a) fails, necessarily (by (b)) $|Y| > \theta$. For each $f \in Y$ let $Y_f = \{h \in Y : h \le_D f\}$, so $|Y_f| < \theta$ hence by the Hajnal free subset theorem for some $Z' \subseteq Z$, $|Z'| = \lambda^+$, and $f_1 \ne f_2 \in Z' \Rightarrow f_1 \notin Y_{f_2}$ so $[f_1 \ne f_2 \in Z' \Rightarrow \neg f_1 <_J f_2]$. But there is no such Z' of cardinality $> 2^\kappa$ ([Sh111, 2.2, p. 264]) so (a) holds. As for (c): choose $f_\alpha \in F$ by induction on α, such that $f_\alpha \in Y \setminus \bigcup_{\beta<\alpha} Y_{f_\beta}$; it exists by cardinality considerations and $\langle f_\alpha : \alpha < \theta\rangle$ is as required (in (c)). $\blacksquare_{6.6A}$

$\blacksquare_{6.6}$

---

* Or strightening clause (i) see the proof of 6.6B

6.6B OBSERVATION: *Let $\kappa < \lambda$ be regular uncountable, $2^\kappa < \mu_i < \lambda$ (for $i < \kappa$), μ_i increasing in i. The following are equivalent:*

(A) *there is $F \subseteq {}^\kappa\lambda$ such that:*
  (i) $|F| = \lambda$,
  (ii) $|\{f \restriction i: f \in F\}| \leq \mu_i$,
  (iii) $[f \neq g \in F \Rightarrow f \neq_{J^{bd}_\kappa} g]$;

(B) *there be a sequence $\langle \lambda_i: i < \kappa\rangle$ such that:*
  (i) $2^\kappa < \lambda_i = \mathrm{cf}(\lambda_i) \leq \mu_i$,
  (ii) $\max \mathrm{pcf}\{\lambda_i: i < \kappa\} = \lambda$,
  (iii) *for $j < \kappa$, $\mu_j \geq \max \mathrm{pcf}\{\lambda_i: i < j\}$;*

(C) *there is an increasing sequence $\langle \mathfrak{a}_i: i < \kappa\rangle$ such that $\lambda \in \mathrm{pcf} \bigcup_{i<\kappa} \mathfrak{a}_i$, $\mathrm{pcf}\, \mathfrak{a}_i \subseteq \mu_i$ (so $\mathrm{Min}(\bigcup_{i<\kappa} \mathfrak{a}_i) > |\bigcup_{i<\kappa} \mathfrak{a}_i|$).*

Proof:

(B)$\Rightarrow$(A): By [Sh355, 3.4].

(A)$\Rightarrow$(B): If $(\forall\theta)[\theta \geq 2^\kappa \Rightarrow \theta^\kappa \leq \theta^+]$ we can directly prove (B) if for a club of $i < \kappa$, $\mu_i > \bigcup_{j<i} \mu_j$, and contradict (A) if this fails. Otherwise every normal filter D on κ is nice (see [Sh386, §1]). Let F exemplify (A).

Let $K = \{(D, g): D$ a normal filter on κ, $g \in {}^\kappa(\lambda+1)$, $\lambda = |\{f \in F: f <_D g\}|\,\}$. Clearly K is not empty (let g be constantly λ) so by [Sh386] we can find $(D, g) \in K$ such that:

$(*)_1$ if $A \subseteq \kappa$, $A \neq \emptyset \bmod D$, $g_1 <_{D+A} g$ then $\lambda > |\{f \in F: f <_{D+A} g_1\}|$.

Let $F^* = \{f \in F: f <_D g\}$, so (as in the proof of 6.6) $|F^*| = \lambda$.

We claim:

$(*)_2$ if $h \in F^*$ then $\{f \in F^*: \neg h \leq_D f\}$ has cardinality $< \lambda$.

[Why? Otherwise for some $h \in F^*$, $F' =: \{f \in F^*: \neg h \leq_D f\}$ has cardinality λ, for $A \subseteq \kappa$ let $F'_A = \{f \in F^*: f \restriction A \leq h \restriction A\}$ so $F' = \bigcup\{F'_A: A \subseteq \kappa, A \neq \emptyset \bmod D\}$, hence for some $A \subseteq \kappa$, $A \neq \emptyset \bmod D$ and $|F'_A| = \lambda$; now $(D + A, h)$ contradicts $(*)_1$].

By $(*)_2$ we can choose by induction on $\alpha < \lambda$, a function $f_\alpha \in F^*$ such that $\bigwedge_{\beta<\alpha} f_\beta <_D f_\alpha$. By [Sh355, 1.2A(3)] $\langle f_\alpha: \alpha < \lambda\rangle$ has an e.u.b. f^*. Let $\lambda_i = \mathrm{cf}(f^*(i))$, clearly $\{i < \kappa: \lambda_i \leq 2^\kappa\} = \emptyset \bmod D$, so without loss of generality $\bigwedge_{i<\kappa} \mathrm{cf}(f^*(i)) > 2^\kappa$ so λ_i is regular $\in (2^\kappa, \lambda]$, and $\lambda = \mathrm{tcf}\left(\prod_{i<\kappa} \lambda_i/D\right)$. Let $J_i = \{A \subseteq i: \max \mathrm{pcf}\{\lambda_j: j < i\} \leq \mu_i\}$; so (remembering (ii) of (A)) we can find $h_i \in \prod_{j<i} f^*(i)$ such that:

$(*)_3$ if $\{j: j < i\} \notin J_i$, then for every $f \in F$, $f \restriction i <_{J_i} h_i$.

Let $h \in \prod_{i<\kappa} f^*(i)$ be defined by: $h(i) = \sup\{h_j(i): j \in (i,\kappa)$ and $\{j: j < i\} \notin J_i\}$. As $\bigwedge_i \mathrm{cf}[f^*(i)] > 2^\kappa$, clearly $h < f^*$ hence by the choice of f^* for some $\alpha(*) < \lambda$ we have: $h <_D f_{\alpha(*)}$ and let $A =: \{i < \kappa: h(i) < f_{\alpha(*)}\}$, so $A \in D$. Define λ'_i as follows: λ'_i is λ_i if $i \in A$, and is $(2^\kappa)^+$ if $i \in \kappa \backslash A$. Now $\langle \lambda'_i: i < \kappa\rangle$ is as required in (B).

(B)⇒(C): Straightforward.

(C)⇒(B): By [Sh371, §1]. $\blacksquare_{6.6B}$

6.6C Claim: *If $F \subseteq {}^\kappa\mathrm{Ord}$, $2^\kappa < \theta = \mathrm{cf}(\theta) \le |F|$ then we can find $g^* \in {}^\kappa\mathrm{Ord}$ and a proper ideal I on κ and $A \subseteq \kappa$, $A \in I$ such that:*

(a) *$\prod_{i<\kappa} g^*(i)/I$ has true cofinality θ, and for each $i \in \kappa \setminus A$ we have $\mathrm{cf}[g^*(i)] > 2^\kappa$,*

(b) *for every $g \in {}^\kappa\mathrm{Ord}$ satisfying $g \restriction A = g^* \restriction A$, $g \restriction (\kappa\backslash A) < g^* \restriction (\kappa\backslash A)$ we can find $f \in F$ such that: $f \restriction A = g^* \restriction A$, $g \restriction (\kappa\backslash A) < f \restriction (\kappa\backslash A) < g^* \restriction (\kappa\backslash A)$.*

Proof: As in [Sh410, 3.7 proof of (A)⇒(B)]. (In short let $f_\alpha \in F$ for $\alpha < \theta$ be distinct, χ large enough, $\langle N_i: i < (2^\kappa)^+\rangle$ as there, $\delta_i =: \sup(\theta \cap N_i)$, $g_i \in {}^\kappa\mathrm{Ord}$, $g_i(\zeta) =: \mathrm{Min}[N \cap \mathrm{Ord} \backslash f_{\delta_i}(\zeta)]$, $A \subseteq \kappa$ and $S \subseteq \{i < (2^\kappa)^+: \mathrm{cf}(i) = \kappa^+\}$ stationary, $[i \in S \Rightarrow g_i = g^*]$, $[\zeta < \alpha \,\&\, i \in S \Rightarrow [f_{\delta_i}(\zeta) = g^*(\zeta) \equiv \zeta \in A]]$ and for some $i(*) < (2^\kappa)^+$, $g^* \in N_{i(*)}$, so $[\zeta \in \kappa \setminus A \Rightarrow \mathrm{cf}\, g^*(\zeta) > 2^\kappa]$.) $\blacksquare_{6.6C}$

6.6D Claim: *Suppose D is a filter on $\theta = \mathrm{cf}(\theta)$, σ-complete, $\theta > |\alpha|^\kappa$ for $\alpha < \sigma$, and for each $\alpha < \theta$, $\bar\beta = \langle \beta^\alpha_\epsilon: \epsilon < \kappa\rangle$ is a sequence of ordinals. Then for every $X \subseteq \theta$, $X \ne \emptyset \bmod D$ there is $\langle \beta^*_\epsilon: \epsilon < \kappa\rangle$ (a sequence of ordinals) and $w \subseteq \kappa$ such that:*

(a) *$\epsilon \in \kappa\backslash w \Rightarrow \sigma \le \mathrm{cf}(\beta^*_\epsilon) \le \theta$,*

(b) *if $\beta'_\epsilon \le \beta^*_\epsilon$ and $[\epsilon \in w \equiv \beta'_\epsilon = \beta^*_\epsilon]$, then $\{\alpha \in X$: for every $\epsilon < \kappa$ we have $\beta'_\epsilon \le \beta^\alpha_\epsilon \le \beta^*_\epsilon$ and $[\epsilon \in w \equiv \beta^\alpha_\epsilon = \beta^*_\epsilon]\} \ne \emptyset \bmod D$.*

Proof: Essentially by the same proof as 6.6C (replacing δ_i by $\mathrm{Min}\{\alpha \in X$: for every $Y \in N_i \cap D$ we have $\alpha \in Y\}$). See more [Sh513, §6]. $\blacksquare_{6.6D}$

6.6E Remark: We can rephrase the conclusion as:

(a) $B =: \{\alpha \in X$: if $\epsilon \in w$ then $\beta^\alpha_\epsilon = \beta^*_\epsilon$, and: if $\epsilon \in \kappa \setminus w$ then β^α_ϵ is $< \beta^*_\epsilon$ but $> \sup\{\beta^*_\zeta: \zeta < \epsilon, \beta^\alpha_\zeta < \beta^*_\epsilon\}\}$ is $\ne \emptyset \bmod D$.

(b) If $\beta'_\epsilon < \beta_\epsilon$ for $\epsilon \in \kappa \setminus w$ then $\{\alpha \in B$: if $\epsilon \in \kappa \setminus w$ then $\beta^\alpha_\epsilon > \beta'_\epsilon\} \ne \emptyset \bmod D$.

(c) $\epsilon \in \kappa \setminus w \Rightarrow \mathrm{cf}(\beta'_\epsilon)$ is $\leq \theta$ but $\geq \sigma$.

6.6F Remark: (1) If $|\mathfrak{a}| < \min(\mathfrak{a})$, $F \subseteq \Pi\mathfrak{a}$, $|F| = \theta = \mathrm{cf}\,\theta \notin \mathrm{pcf}(\mathfrak{a})$ and even $\theta > \sigma = \sup(\theta^+ \cap \mathrm{pcf}(\mathfrak{a}))$ then for some $g \in \Pi\mathfrak{a}$, the set $\{f \in F\colon f < g\}$ is unbounded in θ (or use a σ-complete D as in 6.6E). (This is as $\Pi\mathfrak{a}/J_{<\theta}[\mathfrak{a}]$ is $\min(\mathrm{pcf}(\mathfrak{a}) \setminus \theta)$-directed as the ideal $J_{<\theta}[\mathfrak{a}]$ is generated by $\leq \sigma$ sets; this is discussed in [Sh513, §6].)

6.6G Remark: It is useful to note that 6.6D is useful to use [Sh462, §4, 5.14]: e.g. for if $n < \omega$, $\theta_0 < \theta_1 < \cdots < \theta_n$, satisfying $(*)$ below, for any $\beta'_\epsilon \leq \beta^*_\epsilon$ satisfying $[\epsilon \in w \equiv \beta'_\epsilon < \beta^*_\epsilon]$ we can find $\alpha < \gamma$ in X such that:

$$i \in w \equiv \beta^\alpha_\epsilon = \beta^*_\epsilon,$$

$$\{\epsilon, \zeta\} \subseteq \kappa \setminus w \ \&\ \{\mathrm{cf}(\beta^*_\epsilon), \mathrm{cf}(\beta^*_\zeta)\} \subseteq [\theta_l, \theta_{l+1})) \ \&\ l \text{ even } \Rightarrow \beta^\alpha_\epsilon < \beta^\gamma_\zeta,$$

$$\{\epsilon, \zeta\} \subseteq \kappa \setminus w \ \&\ \{\mathrm{cf}(\beta^*_\zeta), \mathrm{cf}(\beta^*_\zeta)\} \subseteq [\theta_l, \theta_{l+1}) \ \&\ l \text{ odd } \Rightarrow \beta^\gamma_\epsilon < \beta^\alpha_\zeta$$

where

$(*)$ (a) $\epsilon \in \kappa \setminus w \Rightarrow \mathrm{cf}(\beta^*_\epsilon) \in [\theta_0, \theta_n)$, and
(b) $\max\mathrm{pcf}[\{\mathrm{cf}(\beta^*_\epsilon)\colon \epsilon \in \kappa \setminus w\} \cap \theta_l] \leq \theta_l$ (which holds if $\theta_l = \sigma_l^+$, $\sigma_l^\kappa = \sigma_l$ for $l \in \{1, \ldots, n\}$).

6.7 Claim: *For any $\mathfrak{a}$, $|\mathfrak{a}| < \mathrm{Min}(\mathfrak{a})$, we can find $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda\colon \lambda \in \mathfrak{a}\rangle$ such that:*

(α) *$\bar{\mathfrak{b}}$ is a generating sequence, i.e.*

$$\lambda \in \mathfrak{a} \Rightarrow J_{\leq\lambda}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda,$$

(β) *$\bar{\mathfrak{b}}$ is smooth, i.e. for $\theta < \lambda$ in $\mathfrak{a}$,*

$$\theta \in \mathfrak{b}_\lambda \Rightarrow \mathfrak{b}_\theta \subseteq \mathfrak{b}_\lambda,$$

(γ) *$\bar{\mathfrak{b}}$ is closed, i.e. for $\lambda \in \mathrm{pcf}(\mathfrak{a})$ we have $\mathfrak{b}_\lambda = \mathfrak{a} \cap \mathrm{pcf}(\mathfrak{b}_\lambda)$.*

Proof: Let $\langle \mathfrak{b}_\theta[\mathfrak{a}]\colon \theta \in \mathrm{pcf}\,\mathfrak{a}\rangle$ be as in [Sh371, 2.6]. For $\lambda \in \mathfrak{a}$, let $\bar{f}^{\mathfrak{a},\lambda} = \langle f^{\mathfrak{a},\lambda}_\alpha\colon \alpha < \mathfrak{a}\rangle$ be a $<_{J_\lambda[\mathfrak{a}]}$-increasing cofinal sequence of members of $\prod \mathfrak{a}$, satisfying:

$(*)_1$ if $\delta < \lambda$, $|\mathfrak{a}| < \mathrm{cf}(\delta) < \mathrm{Min}\,\mathfrak{a}$ and $\theta \in \mathfrak{a}$ then:

$$f^{\mathfrak{a},\lambda}_\delta(\theta) = \mathrm{Min}\left\{\bigcup_{\alpha \in C} f^{\mathfrak{a},\lambda}_\alpha(\theta)\colon C \text{ a club of } \delta\right\}$$

[exists by [Sh345a, Def. 3.3(2)^b + Fact 3.4(1)]].

Let $\chi = \beth_\omega(\sup \mathfrak{a})^+$, $|\mathfrak{a}| < \kappa = \operatorname{cf}\kappa < \operatorname{Min}\mathfrak{a}$ (without loss of generality there is such κ) and $\bar{N} = \langle N_i : i < \kappa\rangle$ be an increasing continuous sequence of elementary submodels of $(H(\chi), \in, <^*_\chi)$, $N_i \cap \kappa$ an ordinal, $\bar{N} \restriction (i+1) \in N_{i+1}$, $\|N_i\| < \kappa$, and $\mathfrak{a}$, $\langle \bar{f}^{\mathfrak{a},\lambda} : \lambda \in \mathfrak{a}\rangle$ belong to N_0. Let $N_\kappa = \bigcup_{i<\kappa} N_i$. For every $\lambda \in \mathfrak{a}$, for some club E_λ of κ,

$(*)$ $\theta \in \mathfrak{a} \Rightarrow f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)}(\theta) = \bigcup_{\alpha\in E_\lambda} f^{\mathfrak{a},\lambda}_{\sup(N_\alpha\cap\lambda)}(\theta)$.

Let $E = \bigcap_{\lambda\in\mathfrak{a}} E_\lambda$, so E is a club of κ. For any $i < j < \kappa$ let

$$\mathfrak{b}^{i,j}_\lambda = \left\{\theta \in \mathfrak{a} \colon \sup(N_i \cap \theta) < f^{\mathfrak{a},\lambda}_{\sup(N_j\cap\lambda)}(\theta)\right\}.$$

As in the proof of [Sh371, 1.3], possibly shrinking E, we have:

$(*)_2$ for $i < j$ from* E and $\lambda \in \mathfrak{a}$, we have:

(α) $J_{\le\lambda}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}^{i,j}_\lambda$ (hence $\mathfrak{b}^{i,j}_\lambda = \mathfrak{b}_\lambda[\mathfrak{a}] \bmod J_{<\lambda}[\mathfrak{a}]$),

(β) $\mathfrak{b}^{i,j}_\lambda \subseteq \lambda^+ \cap \mathfrak{a}$,

(γ) $\langle \mathfrak{b}^{i,j}_\lambda : \lambda \in \mathfrak{a}\rangle \in N_{j+1}$,

(δ) $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)} \restriction \mathfrak{b}^{i,j}_\lambda = \langle(\theta, \sup(N_\kappa \cap \theta)) \colon \theta \in \mathfrak{b}^{i,j}_\lambda\rangle$,

(ϵ) $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)} \le \langle(\theta, \sup(N_\kappa \cap \theta)) \colon \theta \in \mathfrak{a}\rangle$.

We now define by induction on $\epsilon < |\mathfrak{a}|^+$, for $\lambda \in \mathfrak{a}$ (and $i < j < \kappa$), the set $\mathfrak{b}^{i,j,\epsilon}_\lambda$:

$$\begin{aligned}
\mathfrak{b}^{i,j,0}_\lambda &= \mathfrak{b}^{i,j}_\lambda \\
\mathfrak{b}^{i,j,\epsilon+1}_j &= \mathfrak{b}^{i,j,\epsilon}_\lambda \cup \bigcup\left\{\mathfrak{b}^{i,j,\epsilon}_\theta \colon \theta \in \mathfrak{b}^{i,j,\epsilon}_\lambda\right\} \cup \left\{\theta \in \mathfrak{a} \colon \theta \in \operatorname{pcf}\mathfrak{b}^{i,j,\epsilon}\right\}, \\
\mathfrak{b}^{i,j,\epsilon}_\lambda &= \bigcup_{\zeta<\epsilon} \mathfrak{b}^{i,j,\zeta}_\lambda \quad \text{for } \epsilon < |\mathfrak{a}|^+ \text{ limit.}
\end{aligned}$$

Clearly for $\lambda \in \mathfrak{a}$, $\langle \mathfrak{b}^{i,j,\epsilon}_\lambda : \epsilon < |\mathfrak{a}|^+\rangle$ belongs to N_{j+1} and is a non-decreasing sequence of subsets of $\mathfrak{a}$, hence for some $\epsilon(i,j,\lambda) < |\mathfrak{a}|^+$,

$$\left[\epsilon \in (\epsilon(i,j,\lambda), |\mathfrak{a}|^+) \Rightarrow \mathfrak{b}^{i,j,\epsilon}_\lambda = \mathfrak{b}^{i,j,\epsilon(i,j,\lambda)}_\lambda\right].$$

So letting $\epsilon(i,j) = \sup_{\lambda\in\mathfrak{a}} \epsilon(i,j,\lambda) < |\mathfrak{a}|^+$ we have:

$(*)_3$ $\epsilon(i,j) \le \epsilon < |\mathfrak{a}|^+ \Rightarrow \bigwedge_{\lambda\in\mathfrak{a}} \mathfrak{b}^{i,j,\epsilon(i,j)}_\lambda = \mathfrak{b}^{i,j,\epsilon}_\lambda$.

Which of the properties required from $\langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{a}\rangle$ are satisfied by $\langle \mathfrak{b}^{i,j,\epsilon(i,j)}_\lambda : \lambda \in \mathfrak{a}\rangle$? Note ($\beta$), ($\gamma$) hold by the inductive definition of $\mathfrak{b}^{i,j,\epsilon}_\lambda$ (and the choice of $\epsilon(i,j)$), as for property (α), one half, $J_{\le\lambda}[\mathfrak{a}] \subseteq J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}^{i,j,\epsilon(i,j)}_\lambda$ hold by $(*)_2(\alpha)$ (and $\mathfrak{b}^{i,j}_\lambda = \mathfrak{b}^{i,j,0}_\lambda \subseteq \mathfrak{b}^{i,j,\epsilon(i,j)}_\lambda$), so it is enough to prove (for $\lambda \in \mathfrak{a}$) :

* Actually for any $i < j < \kappa$ clauses (β), (γ), (δ) hold.

$(*)_4$ $\mathfrak{b}_\lambda^{i,j,\epsilon(i,j)} \in J_{\leq\lambda}[\mathfrak{a}]$.

For this end we define by induction on $\epsilon < |\mathfrak{a}|^+$ functions $f_\alpha^{\mathfrak{a},\lambda,\epsilon}$ with domain $\mathfrak{b}_\lambda^{i,j,\epsilon}$ for every $\alpha < \lambda \in \mathfrak{a}$, such that $\zeta < \epsilon \Rightarrow f_\alpha^{\mathfrak{a},\lambda,\zeta} \subseteq f_\alpha^{\mathfrak{a},\lambda,\epsilon}$, so the domain increases with ϵ.

We let $f_\alpha^{\mathfrak{a},\lambda,0} = f_\alpha^{\mathfrak{a},\lambda} \restriction \mathfrak{b}_\lambda^{i,j}$, $f_\alpha^{\mathfrak{a},\lambda,\zeta} = \bigcup_{\zeta<\epsilon} f_\alpha^{\mathfrak{a},\lambda,\zeta}$ for $\epsilon < |\mathfrak{a}|^+$ limit, and $f_\alpha^{\mathfrak{a},\lambda,\epsilon+1}$ is defined by defining each $f_\alpha^{\mathfrak{a},\lambda,\epsilon+1}(\theta)$ as follows:

CASE 1: If $\theta \in \mathfrak{b}_\lambda^{i,j,\epsilon}$ then $f_\alpha^{\mathfrak{a},\lambda,\epsilon}(\theta)$.

CASE 2: If $\mu \in \mathfrak{b}_\lambda^{i,j,\epsilon}$, $\theta \in \mathfrak{b}_\mu^{i,j,\epsilon}$ and not Case 1 and μ minimal under those conditions, then $f_\beta^{\mathfrak{a},\mu,\epsilon}(\theta)$ where we choose $\beta = f_\alpha^{\mathfrak{a},\lambda,\epsilon}(\mu)$.

CASE 3: If $\theta \in \mathfrak{a} \cap \mathrm{pcf}(\mathfrak{b}_\lambda^{i,j,\epsilon})$ and not Case 1 or 2, then

$$\mathrm{Min}\left\{\gamma < \theta\colon f_\alpha^{\mathfrak{a},\lambda,\epsilon} \restriction \mathfrak{b}_\theta[\mathfrak{a}] \leq_{J_{<\theta}[\mathfrak{a}]} f_\gamma^{\mathfrak{a},\theta,\epsilon}\right\}.$$

Now $\langle\langle \mathfrak{b}_\lambda^{i,j,\epsilon}\colon \lambda \in \mathfrak{a}\rangle\colon \epsilon < |\mathfrak{a}|^+\rangle$ can be computed from $\mathfrak{a}$ and $\langle \mathfrak{b}_\lambda^{i,j}\colon \lambda \in \mathfrak{a}\rangle$. But the latter belong* to N_{j+1}, so the former belongs to N_{j+1}, so as also $\langle\langle f_\alpha^{\mathfrak{a},\lambda}\colon \alpha < \lambda\rangle\colon \lambda \in \mathrm{pcf}\,\mathfrak{a}\rangle$ belongs to N_{j+1} we clearly get that

$$\langle\langle\langle f_\alpha^{\mathfrak{a},\lambda,\epsilon}\colon \epsilon < |\mathfrak{a}|^+\rangle\colon \alpha < \lambda\rangle\colon \lambda \in \mathfrak{a}\rangle$$

belongs to N_{j+1}. Next we prove by induction on ϵ that, for $\lambda \in \mathfrak{a}$, we have:

$$\otimes_1 \qquad \theta \in \mathfrak{b}_\lambda^{i,j,\epsilon} \ \&\ \lambda \in \mathfrak{a} \Rightarrow f^{\mathfrak{a},\lambda,\epsilon}_{\sup(N_\kappa\cap\theta)}(\theta) = \sup(N_\kappa \cap \theta).$$

For $\epsilon = 0$ this is by $(*)_2(\delta)$. For ϵ limit, by the induction hypothesis and the definition of $f_\alpha^{\mathfrak{a},\lambda,\epsilon}$. For $\epsilon+1$, we check $f^{\mathfrak{a},\lambda,\epsilon+1}_{\sup(N_\kappa\cap\lambda)}(\theta)$ according to the case in its definition; for Case 1 use the induction hypothesis applied to $f^{\mathfrak{a},\lambda,\epsilon}_{\sup(N_\kappa\cap\lambda)}$. For Case 2 (with μ), by the induction hypothesis applied to $f^{\mathfrak{a},\mu,\epsilon}_{\sup(N_\kappa\cap\mu)}$. Lastly, for Case 3 (with θ) we should note:

(i) $\mathfrak{b}_\lambda^{i,j,\epsilon} \cap \mathfrak{b}_\theta[\mathfrak{a}] \notin J_{<\theta}[\mathfrak{a}]$ (by the case's assumption and $(*)_2(\alpha)$ above),

(ii) $f^{\mathfrak{a},\lambda,\epsilon}_{\sup(N_\kappa\cap\lambda)} \restriction (\mathfrak{b}_\lambda^{i,j,\epsilon} \cap \mathfrak{b}_\theta^{i,j,\epsilon}) \subseteq f^{\mathfrak{a},\theta,\epsilon}_{\sup(N_\kappa\cap\theta)}$ (by the induction hypothesis for ϵ, used concerning λ and θ) hence (by the definition in case 3 and (i) + (ii)),

(iii) $f^{\mathfrak{a},\lambda,\epsilon+1}_{\sup(N_\kappa\cap\lambda)}(\theta) \leq \sup(N_\kappa \cap \theta)$.

---

* As $\langle \mathfrak{b}_\lambda^{i,j,\epsilon} : \lambda \in \mathfrak{a}\rangle : \epsilon|\mathfrak{a}|^+\rangle$ is eventually constant, also each member of the sequence belongs to N_{j+1}.

Now if $\gamma < \sup(N_\kappa \cap \theta)$ then for some $\gamma(1)$, $\gamma < \gamma(1) \in N_\kappa \cap \theta$, so letting $\mathfrak{b} =: \mathfrak{b}^{i,j,\epsilon}_\lambda \cap \mathfrak{b}_\theta[\mathfrak{a}] \cap \mathfrak{b}^{i,j,\epsilon}_\theta$, it belongs to $J_{\leq\theta}[\mathfrak{a}] \setminus J_{<\theta}[\mathfrak{a}]$, we have

$$f^{\mathfrak{a},\theta}_\gamma \restriction \mathfrak{b} <_{J_{<\theta}[\mathfrak{a}]} f^{\mathfrak{a},\theta}_{\gamma(1)} \restriction \mathfrak{b} \leq f^{\mathfrak{a},\theta,\epsilon}_{\sup(N_\kappa \cap \theta)}$$

hence $f^{\mathfrak{a},\lambda,\epsilon+1}_{\sup(N_\kappa\cap\lambda)}(\theta) > \gamma$; as this holds for every $\gamma < \sup(N_\kappa \cap \theta)$ we have obtained

(iv) $f^{\mathfrak{a},\lambda,\epsilon+1}_{\sup(N_\kappa\cap\lambda)}(\theta) \geq \sup(N_\kappa \cap \theta)$;

together we have finished proving the inductive step for $\epsilon + 1$, hence we have proved $\otimes_1$.

This is enough for proving $\mathfrak{b}^{i,j,\epsilon}_\lambda \in J_{\leq\lambda}[\mathfrak{a}]$: Why? If it fails, as $\mathfrak{b}^{i,j,\epsilon}_\lambda \in N_{j+1}$ and $\langle f^{\mathfrak{a},\lambda,\epsilon}_\alpha : \alpha < \lambda\rangle$ belongs to N_{j+1}, there is $g \in \prod \mathfrak{b}^{i,j,\epsilon}_\lambda$ s.t.

$$(*) \qquad \alpha < \lambda \Rightarrow f^{\mathfrak{a},\lambda,\epsilon}_\alpha \restriction \mathfrak{b}^{i,j,\epsilon} < g \bmod J_{\leq\lambda}[\mathfrak{a}].$$

Wlog $g \in N_{j+1}$; by $(*)$, $f^{\mathfrak{a},\lambda,\epsilon}_{\sup(N_\kappa\cap\lambda)} < g \bmod J_{\leq\lambda}[\mathfrak{a}]$. But $g < \langle \sup(N_\kappa \cap \theta) : \theta \in \mathfrak{b}^{i,j,\epsilon}_\lambda\rangle$. Together this contradicts $\oplus_1$!

This ends the proof of 6.7. $\blacksquare_{6.7}$

6.7A Claim: *Assume $|\mathfrak{a}| < \kappa = \mathrm{cf}(\kappa) < \mathrm{Min}(\mathfrak{a})$, σ an infinite ordinal, $|\sigma|^+ < \kappa$. Let $\bar f$, $\bar N = \langle N_i : i < \kappa\rangle$, N_κ be as in the proof of 6.7. Then we can find $\bar i = \langle i_\alpha : \alpha \leq \sigma\rangle$, $\bar{\mathfrak{a}} = \langle \mathfrak{a}_\alpha : \alpha < \sigma\rangle$ and $\langle\langle \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_\beta\rangle : \beta < \sigma\rangle$ such that:*

(a) *$\bar i$ is a strictly increasing continuous sequence of ordinals $< \kappa$,*

(b) *for $\beta < \sigma$ we have $\langle i_\alpha : \alpha \leq \beta\rangle \in N_{i_{\beta+1}}$ (hence* $\langle N_{i_\alpha} : \alpha \leq \beta\rangle \in N_{i_{\beta+1}}$) and $\langle \mathfrak{b}^\gamma_\lambda[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_\gamma$ and $\gamma \leq \beta\rangle \in N_{i_{\beta+1}}$,*

(c) *$\mathfrak{a}_\beta = N_{i_\beta} \cap \mathrm{pcf}(\mathfrak{a})$, so $\mathfrak{a}_\beta$ is increasing continuous in β, $\mathfrak{a} \subseteq \mathfrak{a}_\beta \subseteq \mathrm{pcf}\,\mathfrak{a}$, $|\mathfrak{a}_\beta| < \kappa$,*

(d) *$\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}] \subseteq \mathfrak{a}_\beta$ (for $\lambda \in \mathfrak{a}_\beta$),*

(e) *$J_{\leq\lambda}[\mathfrak{a}_\beta] = J_{<\lambda}[\mathfrak{a}_\beta] + \mathfrak{b}^\beta_\lambda[\mathfrak{a}]$ (so $\lambda \in \mathfrak{b}_\lambda[\mathfrak{a}]$ and $\mathfrak{b}_\lambda[\mathfrak{a}] \subseteq \lambda^+$),*

(f) *if $\mu < \lambda$ are in $\mathfrak{a}_\beta$ and $\mu \in \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$ then $\mathfrak{b}^\beta_\mu[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$ (i.e. smoothness),*

(g) *$\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}] = \mathfrak{a}_\beta \cap \mathrm{pcf}\,\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$ (i.e. closedness),*

(h) *if $\mathfrak{c} \subseteq \mathfrak{a}_\beta$, $\beta < \sigma$, $\mathfrak{c} \in N_{i_{\beta+1}}$ then for some finite $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1} \cap \mathrm{pcf}(\mathfrak{c})$, we have $\mathfrak{c} \subseteq \bigcup_{\mu\in\mathfrak{d}} \mathfrak{b}^{\beta+1}_\mu[\bar{\mathfrak{a}}]$; more generally,***

(h)$^+$ *if $\mathfrak{c} \subseteq \mathfrak{a}_\beta$, $\beta < \sigma$, $\mathfrak{c} \in N_{i_{\beta+1}}$, $\theta = \mathrm{cf}(\theta) \in N_{i_{\beta+1}}$, then for some $\mathfrak{d} \in N_{i_{\beta+1}}$, $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1} \cap \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c})$ we have $\mathfrak{c} \subseteq \bigcup_{\mu\in\mathfrak{d}} \mathfrak{b}^{\beta+1}_\mu[\bar{\mathfrak{a}}]$ and $|\mathfrak{d}| < \theta$,*

* We can get $\bar i \restriction (\beta + 1) \in N_{i_\beta+1}$ if κ succesor of regular and $\bar C$ a square later.

** If in (h)$^+$, $\theta = \aleph_0$, we get (h).

(i) $\mathfrak{b}^{\beta}_{\lambda}[\bar{\mathfrak{a}}]$ *increases with* β.

This will be proved below.

6.7B Claim: *In 6.7A we can also have:*

(1) *if we let* $\mathfrak{b}_\lambda[\bar{\mathfrak{a}}] = \mathfrak{b}^{\sigma}_{\lambda}[\mathfrak{a}] = \bigcup_{\beta<\sigma} \mathfrak{b}^{\beta}_{\lambda}[\bar{\mathfrak{a}}]$, $\mathfrak{a}_\sigma = \bigcup_{\beta<\sigma} \mathfrak{a}_\beta$ *then also for* $\beta = \sigma$ *we have* (b) *(use* $N_{i_\beta+1}$*)*, (c), (d), (f), (i).

(2) *If* $\sigma = \mathrm{cf}(\sigma) > |\mathfrak{a}|$ *then for* $\beta = \sigma$ *also* (e), (g).

(3) *If* $\mathrm{cf}(\sigma) > |\mathfrak{a}|$, $\mathfrak{c} \in N_{i_\sigma}$, $\mathfrak{c} \subseteq \mathfrak{a}_\sigma$ *(hence* $|\mathfrak{c}| < \mathrm{Min}(\mathfrak{c})$ *and* $\mathfrak{c} \subseteq \mathfrak{a}_\sigma$*), then for some finite* $\mathfrak{d} \subseteq (\mathrm{pcf}\,\mathfrak{c}) \cap \mathfrak{a}_\sigma$ *we have* $\mathfrak{c} \subseteq \bigcup_{\mu\in\mathfrak{d}} \mathfrak{b}_\mu[\bar{\mathfrak{a}}]$. *Similarly for* θ*-complete,* $\theta < \mathrm{cf}(\sigma)$ *(i.e. we have clauses* (h), (h)$^+$ *for* $\beta = \sigma$*).*

(4) *We can have continuity in* $\delta \le \sigma$ *when* $\mathrm{cf}(\delta) > |\mathfrak{a}|$, *i.e.* $\mathfrak{b}^{\delta}_{\lambda} = \bigcup_{\beta<\delta} \mathfrak{b}^{\beta}_{\lambda}$.

6.7C Remark:

(1) If we want to use length κ, use $\bar{N}$ as produced in [Sh420, 2.6] so $\sigma = \kappa$.

(2) Concerning 6.7B, in 6.7C(1) for a club E of $\sigma = \kappa$, we have $\alpha \in E \Rightarrow \mathfrak{b}^{\alpha}_{\lambda}[\bar{\mathfrak{a}}] = \mathfrak{b}_\lambda[\bar{\mathfrak{a}}] \cap \mathfrak{a}_\alpha$.

(3) We can also use 6.7 (6.7A, 6.7B) to give an alternative proof of part of the localization theorems similar to the one given in the Spring '89 lectures. For example:

(3A) If $|\mathfrak{a}| < \theta = \mathrm{cf}\,\theta < \mathrm{Min}(\mathfrak{a})$, for no $\lambda_i \in \mathrm{pcf}\,\mathfrak{a}$ $(i < \theta)$ $\alpha < \theta$, do we have $\bigwedge_{\alpha<\theta}[\lambda_\alpha > \max\mathrm{pcf}\{\lambda_i : i < \alpha\}]$.

(3B) if $|\mathfrak{a}| < \mathrm{Min}(\mathfrak{a})$, $|\mathfrak{b}| < \mathrm{Min}\,\mathfrak{b}$, $\mathfrak{b} \subseteq \mathrm{pcf}(\mathfrak{a})$, $\lambda \in \mathrm{pcf}(\mathfrak{a})$, then for some $\mathfrak{c} \subseteq \mathfrak{b}$ we have $|\mathfrak{c}| \le |\mathfrak{a}|$ and $\lambda \in \mathrm{pcf}(\mathfrak{c})$.

Proof of (3A) from 6.7C(3): Without loss of generality $\mathrm{Min}\,\mathfrak{a} > \theta^{+3}$, let $\kappa = \theta^{+2}$, let $\bar{N}$, N_κ, $\bar{\mathfrak{a}}$, $\mathfrak{b}$ (as a function), $\langle i_\alpha : \alpha \le \sigma =: |\mathfrak{a}|^+\rangle$ be as in 6.7A but also $\langle \lambda_i : i < \theta\rangle \in N_0$. So for $j < \theta$, $\mathfrak{c}_j =: \{\lambda_i : i < j\} \in N_0$ (and $\mathfrak{c}_j \subseteq \mathfrak{a}_0$) hence (by clause (h) of 6.7A), for some finite $\mathfrak{d}_j \subseteq \mathfrak{a}_1 \cap \mathrm{pcf}\,\mathfrak{c}_j = N_{i_1} \cap \mathrm{pcf}\,\mathfrak{a} \cap \mathrm{pcf}\,\mathfrak{c}_j$ we have $\mathfrak{c}_j \subseteq \bigcup_{\lambda\in\mathfrak{d}_j} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]$. Assume $j(1) < j(2) < \theta$. Now if $\mu \in \mathfrak{a} \cap \bigcup_{\lambda\in\mathfrak{d}_{j(1)}} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]$ then for some $\mu_0 \in \mathfrak{d}_{j(1)}$ we have $\mu \in \mathfrak{b}^1_{\mu_0}[\bar{\mathfrak{a}}]$; now $\mu_0 \in \mathfrak{d}_{j(1)} \subseteq \mathrm{pcf}(\mathfrak{c}_{j(1)}) \subseteq \mathrm{pcf}(\mathfrak{c}_{j(2)}) \subseteq \mathrm{pcf}\left(\bigcup_{\lambda\in\mathfrak{d}_{j(2)}} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]\right) = \bigcup_{\lambda\in\mathfrak{d}_{j(2)}} \mathrm{pcf}(\mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}])$ hence (by clause (g) of 6.7A as $\mu_0 \in \mathfrak{d}_{j(0)} \subseteq N_1$) for some $\mu_1 \in \mathfrak{d}_{j(2)}$, $\mu_0 \in \mathfrak{b}^1_{\mu_1}[\bar{\mathfrak{a}}]$. So by clause (f) of 6.7A we have $\mathfrak{b}^1_{\mu_0}[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^1_{\mu_1}[\bar{\mathfrak{a}}]$ so remembering $\mu \in \mathfrak{b}^1_{\mu_0}[\bar{\mathfrak{a}}]$, we have $\mu \in \mathfrak{b}^1_{\mu_1}[\bar{\mathfrak{a}}]$. Remembering μ was any member of $\mathfrak{a} \cap \bigcup_{\lambda\in\mathfrak{d}_{j(1)}} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]$, we have $\mathfrak{a} \cap \bigcup_{\lambda\in\mathfrak{d}j(1)} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}] \subseteq \mathfrak{a} \cap \bigcup_{\lambda\in\mathfrak{d}j(2)} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]$ (holds without "$\mathfrak{a}\cap$" but not used). So $\langle \mathfrak{a} \cap \bigcup_{\lambda\in\mathfrak{d}_j} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}] : j < \theta\rangle$ is a non-decreasing sequence of subsets of $\mathfrak{a}$, but $\mathrm{cf}(\theta) > |\mathfrak{a}|$, so the sequence is eventually constant, say for $j \geq j(*)$. But

$$\begin{aligned}\max\operatorname{pcf}\left(\mathfrak{a}\cap\bigcup_{\lambda\in\mathfrak{d}_j}\mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]\right) &\leq \max\operatorname{pcf}\left(\bigcup_{\lambda\in\mathfrak{d}_j}\mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]\right)\\ &= \max_{\lambda\in\mathfrak{d}_j}\left(\max\operatorname{pcf}(\mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}])\right)\\ &= \max_{\lambda\in\mathfrak{d}_j}\lambda \leq \max\operatorname{pcf}\{\lambda_i\colon i<j\} < \lambda_j\\ &= \max\operatorname{pcf}\left(\mathfrak{a}\cap\bigcup_{\lambda\in\mathfrak{d}_{j+1}}\mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]\right)\end{aligned}$$

(last equality as $\mathfrak{b}_{\lambda_j}[\mathfrak{a}] \subseteq \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}] \bmod J_{<\lambda}[\mathfrak{a}_1]$). Contradiction. $\blacksquare_{6.7C}$

Proof of 6.7C(3B) (like [Sh371, §3]): Included for completeness. If this fails choose a counterexample $(\mathfrak{a},\mathfrak{b},\lambda)$ with $|\mathfrak{b}|$ minimal, and among those with $\max\operatorname{pcf}(\mathfrak{b})$ minimal and among those with $\bigcup\{\mu^+\colon \mu\in\lambda\cap\operatorname{pcf}(\mathfrak{b})\}$ minimal. So $\max\operatorname{pcf}(\mathfrak{b})=\lambda$, and $\mu=\sup[\lambda\cap\operatorname{pcf}(\mathfrak{a})]$ is not in $\operatorname{pcf}(\mathfrak{b})$ or $\mu=\lambda$. Try to choose by induction on $i<|\mathfrak{a}|^+$, $\lambda_i\in\lambda\cap\operatorname{pcf}(\mathfrak{b})$, $\lambda_i>\max\operatorname{pcf}\{\lambda_j\colon j<i\}$, by 6.7C(3A), we will be stuck at some i, and by the previous sentence (and choice of $(\mathfrak{a},\mathfrak{b},\lambda)$, i is limit, so $\operatorname{pcf}(\{\lambda_j\colon j<i\})\not\subseteq\lambda$ but it is $\subseteq\operatorname{pcf}(\mathfrak{b})\subseteq\lambda^+$, so $\lambda=\max\operatorname{pcf}\{\lambda_j\colon j<i\}$. For each j, by the minimality condition for some $\mathfrak{b}_j\subseteq\mathfrak{b}$, we have $|\mathfrak{b}_j|\leq|\mathfrak{a}|$, $\lambda_j\in\operatorname{pcf}(\mathfrak{b}_j)$. So $\lambda\in\operatorname{pcf}\{\lambda_j\colon j<i\}\subseteq\operatorname{pcf}(\bigcup_{j<i}\mathfrak{b}_j)$ but $\bigcup_{j<i}\mathfrak{b}_j$ is a subset of $\mathfrak{b}$ of cardinality $\leq|i|\times|\mathfrak{a}|=|\mathfrak{a}|$.

6.7D Proof of 6.7A: Let $\langle\langle f^{\mathfrak{a},\lambda}_\alpha\colon \alpha<\lambda\rangle\colon \lambda\in\operatorname{pcf}\mathfrak{a}\rangle$ be chosen as in the proof of 6.7. For $\zeta<\kappa$ we define $\mathfrak{a}^\zeta =: N_\zeta\cap\operatorname{pcf}\mathfrak{a}$; we also define ${}^\zeta\bar{f}$ as $\langle\langle f^{\mathfrak{a}^\zeta,\lambda}_\alpha\colon \alpha<\lambda\rangle\colon\lambda\in\operatorname{pcf}\mathfrak{a}\rangle$ where $f^{\mathfrak{a}^\zeta,\lambda}_\alpha\in\prod\mathfrak{a}^\zeta$ is defined as follows:

(a) if $\theta\in\mathfrak{a}$, $f^{\mathfrak{a}^\zeta,\lambda}_\alpha(\theta)=f^{\mathfrak{a},\lambda}_\alpha(\theta)$,

(b) if $\theta\in\mathfrak{a}^\zeta\backslash\mathfrak{a}$ and $\operatorname{cf}(\alpha)\notin(|\mathfrak{a}^\zeta|,\operatorname{Min}\mathfrak{a})$, then

$$f^{\mathfrak{a}^\zeta,\lambda}_\alpha(\theta)=\operatorname{Min}\left\{\gamma<\theta\colon f^{\mathfrak{a},\lambda}_\alpha\restriction\mathfrak{b}_\theta[\mathfrak{a}]\leq_{J_{<\theta}[\mathfrak{b}_\theta[\mathfrak{a}]]}f^{\mathfrak{a},\theta}_\gamma\restriction\mathfrak{b}_\theta[\mathfrak{a}]\right\},$$

(c) if $\theta\in\mathfrak{a}^\zeta\backslash\mathfrak{a}$ and $\operatorname{cf}(\alpha)\in(|\mathfrak{a}^\zeta|,\operatorname{Min}\mathfrak{a})$, define $f^{\mathfrak{a}^\zeta,\lambda}_\alpha(\theta)$ so as to satisfy $(*)_1$ in the proof of 6.7.

Now ${}^\zeta\bar{f}$ is legitimate except that we have only

$$\beta<\gamma<\lambda\in\operatorname{pcf}\mathfrak{a}\Rightarrow f^{\mathfrak{a}^\zeta,\lambda}_\beta\leq f^{\mathfrak{a}^\zeta,\lambda}_\gamma \bmod J_{<\lambda}[\mathfrak{a}^\zeta]$$

(instead of strict inequality) and $\bigwedge_{\beta<\lambda} \bigvee_{\gamma<\lambda} \left[f_\beta^{\mathfrak{a}^\zeta,\lambda} < f_\gamma^{\mathfrak{a}^\zeta,\lambda} \bmod J_{<\lambda}[\mathfrak{a}^\zeta]\right]$, but this suffices. (The first statement is actually proved in [Sh371, 3.2A], the second in [Sh371, 3.2B]; by it also ${}^\zeta \bar{f}$ is cofinal in the required sense.)

For every $\zeta < \kappa$ we can apply 6.7 with $(N_\zeta \cap \mathrm{pcf}\, \mathfrak{a})$, ${}^\zeta \bar{f}$ and $\langle N_{\zeta+1+i}: i < \kappa\rangle$ here standing for $\mathfrak{a}$, $\bar{f}$, $\bar{N}$ there. In the proof of 6.7 get a club E_ζ of κ (so any $i < j$ from E_ζ are O.K.). Now we can define for $\zeta < \kappa$ and $i < j$ in E_ζ, ${}^\zeta\mathfrak{b}_\lambda^{i,j}$ and $\langle {}^\zeta\mathfrak{b}_\lambda^{i,j,\epsilon}: \epsilon < |\mathfrak{a}^\zeta|^+\rangle$, $\langle \epsilon^\zeta(i,j,\lambda): \lambda \in \mathfrak{a}^\zeta\rangle$, $\epsilon^\zeta(i,j)$, as well as in the proof of 6.7. Let:

$$E = \{i < \kappa: i \quad \text{is a limit ordinal } (\forall j < i)(j+j < i \,\&\, j\times j < i) \text{ and } \bigwedge_{j<i} i \in E_j\ \}.$$

So by [Sh420, §1] we can find $\bar{C} = \langle C_\delta: \delta \in S\rangle$, $S \subseteq \{\delta < \kappa: \mathrm{cf}\,\delta = \mathrm{cf}\,\sigma\}$ stationary, C_δ a club of δ, $\mathrm{otp}\, C_\delta = \omega^2\sigma$ such that:

(1) for each $\alpha < \lambda$, $\{C_\delta \cap \alpha: \alpha \in \mathrm{nacc}(C_\delta)\}$ has cardinality $< \kappa$,* and

(2) for every club E' of θ for stationarily many $\delta \in S$, $C_\delta \subseteq E'$.

Without loss of generality $\bar{C} \in N_0$. For some δ^*, $C_{\delta^*} \subseteq E$, and let $\{j_\zeta: \zeta \leq \omega^2\sigma\}$ enumerate $C_{\delta^*} \cup \{\delta^*\}$. So $\langle j_\zeta: \zeta \leq \omega^2\sigma\rangle$ is a strictly increasing continuous sequence of ordinals from $E \subseteq \kappa$ such that $\langle j_\epsilon: \epsilon \leq \zeta\rangle \in N_{j_{\zeta+1}}$. Let $j(\zeta) = j_\zeta$, $i(\zeta) = i_\zeta =: j_{\omega^2(1+\zeta)}$, $\mathfrak{a}_\zeta = N_{i_\zeta} \cap \mathrm{pcf}\,\mathfrak{a}$, and $\bar{\mathfrak{a}} =: \langle \mathfrak{a}_\zeta: \zeta < \sigma\rangle$, $\mathfrak{b}_\lambda^\zeta[\bar{\mathfrak{a}}] =: {}^{i(\zeta)}\mathfrak{b}_\lambda^{j(\omega^2\zeta+1),j(\omega^2\zeta+2),\epsilon^\zeta(j(\omega^2\zeta+1),j(\omega^2\zeta+2))}$. Most of the requirements follow immediately, as

($*$) for each $\zeta < \sigma$, we have $\mathfrak{a}_\zeta$, $\langle \mathfrak{b}_\lambda^\zeta[\bar{\mathfrak{a}}]: \lambda \in \mathfrak{a}_\zeta\rangle$ are as in 6.7 and belong to $N_{i_\beta+3} \subseteq N_{i_{\beta+1}}$.

We are left (for proving 6.7A) with proving (h)$^+$ and (i) (remember (h) is a special case of (h)$^+$ choosing $\theta = \aleph_0$).

For proving clause (i) note that for $\zeta < \xi < \kappa$, $f_\alpha^{\mathfrak{a}^\zeta,\lambda} \subseteq f_\alpha^{\mathfrak{a}^\xi,\lambda}$ hence ${}^\zeta\mathfrak{b}_\lambda^{i,j} \subseteq {}^\xi\mathfrak{b}_\lambda^{i,j}$. Now we can prove by induction on ϵ that ${}^\zeta\mathfrak{b}_\lambda^{i,j,\epsilon} \subseteq {}^\xi\mathfrak{b}_\lambda^{i,j,\epsilon}$ for every $\lambda \in \mathfrak{a}_\zeta$ (check the definition after $(*)_2$ in the proof of 6.7) and the conclusion follows.

Instead of proving (h)$^+$ we prove an apparently weaker version (h)$'$ below, and then note that $\bar{i}' = \langle i_{\omega^2\zeta}: \zeta < \sigma\rangle$, $\bar{\mathfrak{a}}' = \langle \mathfrak{a}_{\omega^2\zeta}: \zeta < \sigma\rangle$, $\langle N_{i(\omega^2\zeta)}: \zeta < \sigma\rangle$, $\langle \mathfrak{b}_\lambda^{\omega^2\zeta}[\bar{\mathfrak{a}}']: \zeta < \sigma, \lambda \in \mathfrak{a}'_\zeta = \mathfrak{a}_{\omega^2\zeta}\rangle$ will exemplify the conclusion** where

(h)$'$ if $\mathfrak{c} \subseteq \mathfrak{a}_\beta$, $\beta < \sigma$, $\mathfrak{c} \in N_{i_{\beta+1}}$, $\theta = \mathrm{cf}(\theta) \in N_{i_{\beta+1}}$ *then* for some $\mathfrak{d} \in N_{i_{\beta+\omega+1}+1}$, $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+\omega} \cap \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ we have $\mathfrak{c} \subseteq \bigcup_{\mu\in\mathfrak{d}} \mathfrak{b}_\mu^{\beta+\omega}[\bar{\mathfrak{a}}]$ and $|\mathfrak{d}| < \theta$.

---

* If κ is successor of regular, then we can get $[\gamma \in C_\alpha \cap C_\beta \Rightarrow C_\alpha \cap \gamma = C_\beta \cap \gamma]$.

** Assuming $\sigma > \aleph_0$ hence, $\omega^2\sigma = \sigma$ for notational simplicity.

Proof of (h)′: So let θ, β, $\mathfrak{c}$ be given; let $\langle \mathfrak{b}_\mu[\mathfrak{a}]: \mu \in \mathrm{pcf}\, \mathfrak{c}\rangle (\in N_{i_{\beta+1}})$ be a generating sequence. We define by induction on $n < \omega$, A_n, $\langle \mathfrak{c}_\eta, \lambda_\eta: \eta \in A_n\rangle$ such that:

(a) $A_0 = \{\langle\rangle\}$, $\mathfrak{c}_{\langle\rangle} = \mathfrak{c}$, $\lambda_{\langle\rangle} = \max \mathrm{pcf}\, \mathfrak{c}$,

(b) $A_n \subseteq {}^n\theta$, $|A_n| < \theta$,

(c) if $\eta \in A_{n+1}$ then $\eta \restriction n \in A_n$, $\mathfrak{c}_\eta \subseteq \mathfrak{c}_{\eta \restriction n}$, $\lambda_\eta < \lambda_{\eta\restriction n}$ and $\lambda_\eta = \max \mathrm{pcf}(\mathfrak{c}_\eta)$,

(d) A_n, $\langle \mathfrak{c}_\eta, \lambda_\eta: \eta \in A_n\rangle$ belongs to $N_{i_{\beta+1+n}}$ hence $\lambda_\eta \in N_{i_{\beta+1+n}}$,

(e) if $\eta \in A_n$ and $\lambda_\eta \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c}_\eta)$ and $\mathfrak{c}_\eta \not\subseteq \mathfrak{b}^{\beta+1+n}_{\lambda_\eta}[\bar{\mathfrak{a}}]$ then $(\forall \nu)[\nu \in A_{n+1} \,\&\, \eta \subseteq \nu \Leftrightarrow \nu = \eta\hat{\ }\langle 0\rangle]$ and $\mathfrak{c}_{\eta\hat{\ }\langle 0\rangle} = \mathfrak{c}_\eta \setminus \mathfrak{b}^{\beta+1+n}_{\lambda_\eta}[\bar{\mathfrak{a}}]$ (so $\lambda_{\eta\hat{\ }\langle 0\rangle} = \max \mathrm{pcf}\, \mathfrak{c}_{\eta\hat{\ }\langle 0\rangle} < \lambda_\eta = \max\mathrm{pcf}\, \mathfrak{c}_\eta$),

(f) if $\eta \in A_n$ and $\lambda_\eta \notin \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c}_\eta)$ then

$$\mathfrak{c}_\eta = \bigcup \left\{ \mathfrak{b}_{\lambda_{\eta\hat{\ }\langle i\rangle}}[\mathfrak{c}]: i < i_n < \theta, \eta\hat{\ }\langle i\rangle \in A_{n+1}\right\},$$

and if $\nu = \eta\hat{\ }\langle i\rangle \in A_{n+1}$ then $\mathfrak{c}_\nu = \mathfrak{b}_{\lambda_\nu}[\mathfrak{c}]$,

(g) if $\eta \in A_n$, and $\lambda_\eta \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c}_\eta)$ but $\mathfrak{c}_\eta \subseteq \mathfrak{b}^{\beta+1-n}_{\lambda_n}[\bar{\mathfrak{a}}]$, then $\neg(\exists\nu)[\eta \triangleleft \nu \in A_{n+1}]$.

There is no problem to carry the definition (we use 6.7F(1) below*, the point is that $\mathfrak{c} \in N_{i_{\beta+1+n}}$ implies $\langle \mathfrak{b}_\lambda[\mathfrak{c}]: \lambda \in \mathrm{pcf}_\theta[\mathfrak{c}]\rangle \in N_{i_{\beta+1+n}}$ and as there is $\mathfrak{d}$ as in 6.7F(1), there is one in $N_{i_{\beta+1+n+1}}$ so $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1+n+1}$). Now let

$$\mathfrak{d}_n =: \left\{ \lambda_\eta: \eta \in A_n \text{ and } \lambda_\eta \in \underset{\theta-\mathrm{complete}}{\mathrm{pcf}}(\mathfrak{c}_\eta) \text{ and } \mathfrak{c}_\eta \subseteq \mathfrak{b}^{\beta+1+n}_{\lambda_\eta}[\mathfrak{a}]\right\}$$

and $\mathfrak{d} =: \bigcup_{n<\omega} \mathfrak{d}_n$; we shall show that it is as required.

The main point is $\mathfrak{c} \subseteq \bigcup_{\lambda \in \mathfrak{d}} \mathfrak{b}^{\beta+\omega}_\lambda[\bar{\mathfrak{a}}]$; note that

$$\left[\lambda_\eta \in \mathfrak{d}, \eta \in A_n \Rightarrow \mathfrak{b}^{\beta+1+n}_{\lambda_\eta}[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^{\beta+\omega}_{\lambda_\eta}[\bar{\mathfrak{a}}]\right]$$

hence it suffices to show $\mathfrak{c} \subseteq \bigcup_{n<\omega}\bigcup_{\lambda\in\mathfrak{d}_n} \mathfrak{b}^{\beta+1+n}_\lambda[\bar{\mathfrak{a}}]$, so assume $\theta \in \mathfrak{c} \setminus \bigcup_{n<\omega}\bigcup_{\lambda\in\mathfrak{d}_n} \mathfrak{b}^{\beta+1+n}_\lambda[\bar{\mathfrak{a}}]$, and we choose by induction on n, $\eta_n \in A_n$ such that $\eta_0 = <>$, $\eta_{n+1} \restriction n = \eta_n$ and $\theta \in \mathfrak{c}_\eta$; by clauses (e) + (f) above this is possible and $\langle \max \mathrm{pcf}\, \mathfrak{c}_{\eta_n}: n < \omega\rangle$ is strictly decreasing, contradiction.

The minor point is $|\mathfrak{d}| < \theta$; if $\theta > \aleph_0$ note that $\bigwedge_n |A_n| < \theta$ and $\theta = \mathrm{cf}(\theta)$ so $|\mathfrak{d}| \leq |\bigcup_n A_n| < \theta + \aleph_1 = \theta$.

---

* No vicious circle; 6.7F(1) does not depend on 6.7B.

If $\theta = \aleph_0$ (i.e. clause (h)) we should have $\bigcup_n A_n$ finite; the proof is as above noting the clause (f) is vacuous now. So $\bigwedge_n |A_n| = 1$ and $\bigvee_n A_n = \emptyset$, so $\bigcup_n A_n$ is finite. Another minor point is $\mathfrak{d} \in N_{i_{\beta+\omega+1}}$; this holds as the construction is unique from $\langle N_j : j < i_{\beta+\omega}\rangle$, $\langle i_j : j \le \beta+\omega\rangle$, $\langle(\mathfrak{a}_{i(\zeta)}, \langle \mathfrak{b}^\zeta_\lambda : \lambda \in \mathfrak{a}_{i(\zeta)}\rangle) : \zeta \le \beta+\omega\rangle$; no "outside" information is used so $\langle (A_n, \langle (c_\eta, \lambda_\eta) : \eta \in A_n\rangle) : n < \omega\rangle \in N_{i_{\beta+\omega+1}}$, so (using a choice function) really $\mathfrak{d} \in N_{i_{\beta+\omega+1}}$. $\blacksquare_{6.7A}$

6.7E Proof of 6.7B: Let $\mathfrak{b}_\lambda[\bar{\mathfrak{a}}] = \mathfrak{b}^\sigma_\lambda = \bigcup_{\beta<\sigma} \mathfrak{b}^\beta_\lambda[\mathfrak{a}_\beta]$ and $\mathfrak{a}_\sigma = \bigcup_{\zeta<\sigma} \mathfrak{a}_\zeta$. Part (1) is straightforward. For part (2), for clause (g), for $\beta = \sigma$, the inclusion "$\subseteq$" is straightforward; so assume $\mu \in \mathfrak{a}_\beta \cap \operatorname{pcf} \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$. Then by 6.7A(c) for some $\beta_0 < \beta$, we have $\mu \in \mathfrak{a}_{\beta_0}$, and by 6.7C(3B) (which depends on 6.7A only) for some $\beta_1 < \beta$, $\mu \in \operatorname{pcf} \mathfrak{b}^{\beta_1}_\lambda[\bar{\mathfrak{a}}]$; by monotonicity wlog $\beta_0 = \beta_1$, by clause (g) of 6.7A applied to β_0, $\mu \in \mathfrak{b}^{\beta_0}_\lambda[\bar{\mathfrak{a}}]$. Hence by clause (i) of 6.7A, $\mu \in \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$, thus proving the other inclusion.

The proof of clause (e) (for 6.7B(2)) is similar, and also 6.7B(3). For 6.7(B)(4) for $\delta < \sigma$, $\operatorname{cf}(\delta) > |\mathfrak{a}|$ redefine $\mathfrak{b}^\delta_\lambda[\bar{\mathfrak{a}}]$ as $\bigcup_{\beta<\delta} \mathfrak{b}^{\beta+1}_\lambda[\mathfrak{a}]$. $\blacksquare_{6.7B}$

6.7F Claim: *Let θ be regular.*

(0) *If $\alpha < \theta$, $\operatorname{pcf}_{\theta-\text{complete}}\left(\bigcup_{i<\alpha} \mathfrak{a}_i\right) = \bigcup_{i<\alpha} \operatorname{pcf}_{\theta-\text{complete}}(\mathfrak{a}_i)$.*

(1) *If $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \operatorname{pcf}\mathfrak{a}\rangle$ is a generating sequence for $\mathfrak{a}$, $\mathfrak{c} \subseteq \mathfrak{a}$, then for some $\mathfrak{d} \subseteq \operatorname{pcf}_{\theta-\text{complete}}(\mathfrak{c})$ we have: $|\mathfrak{d}| < \theta$ and $\mathfrak{c} \subseteq \bigcup_{\theta\in\mathfrak{a}} \mathfrak{b}_\theta[\mathfrak{a}]$.*

(2) *If $|\mathfrak{a} \cup \mathfrak{c}| < \operatorname{Min}\mathfrak{a}$, $\mathfrak{c} \subseteq \operatorname{pcf}_{\theta-\text{complete}}(\mathfrak{a})$, $\lambda \in \operatorname{pcf}_{\theta-\text{complete}}(\mathfrak{c})$ then $\lambda \in \operatorname{pcf}_{\theta-\text{complete}}(\mathfrak{a})$.*

(3) *In (2) we can weaken $|\mathfrak{a} \cup \mathfrak{c}| < \operatorname{Min}\mathfrak{a}$ to $|\mathfrak{a}| < \operatorname{Min}\mathfrak{a}$, $|\mathfrak{c}| < \operatorname{Min}\mathfrak{c}$.*

(4) *We cannot find $\lambda_\alpha \in \operatorname{pcf}_{\theta-\text{complete}}(\mathfrak{a})$ for $\alpha < |\mathfrak{a}|^+$ such that $\lambda_i > \sup \operatorname{pcf}_{\theta-\text{complete}}(\{\lambda_j : j < i\})$.*

(5) *Assume $\theta \le |\mathfrak{a}|$, $\mathfrak{c} \subseteq \operatorname{pcf}_{\theta-\text{complete}}\mathfrak{a}$ (and $|\mathfrak{c}| < \operatorname{Min}\ \mathfrak{c}$; of course $|\mathfrak{a}| < \operatorname{Min}\mathfrak{a}$). If $\lambda \in \operatorname{pcf}_{\theta-\text{complete}}(\mathfrak{c})$ then for some $\mathfrak{d} \subseteq \mathfrak{c}$ we have $|\mathfrak{d}| \le |\mathfrak{a}|$ and $\lambda \in \operatorname{pcf}_{\theta-\text{complete}}(\mathfrak{d})$.*

Proof: (0) and (1): Check.

(2) See [Sh345b, 1.10–1.12].

(3) Similarly.

(4) If $\theta = \aleph_0$ we already know it (e.g. 6.7C(3A)), so assume $\theta > \aleph_0$ and, without loss of generality, θ is regular $\le |\mathfrak{a}|$. We use 6.7A with $\{\theta, \langle \lambda_i : i < |\mathfrak{a}|^+\rangle\} \in N_0$, $\sigma = |\mathfrak{a}|^+$, $\kappa = |\mathfrak{a}|^{+3}$ where, without loss of generality, $\kappa < \operatorname{Min}(\mathfrak{a})$. For each $\alpha < |\mathfrak{a}|^+$ by $(h)^+$ of 6.7A there is $\mathfrak{d}_\alpha \in N_{i_1}$, $\mathfrak{d}_\alpha \subseteq \operatorname{pcf}_{\theta-\text{complete}}(\{\lambda_i : i < \alpha\})$, $|\mathfrak{d}_\alpha| < \theta$

such that $\{\lambda_i: i < \alpha\} \subseteq \bigcup_{\theta\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\theta[\bar{\mathfrak{a}}]$; hence by clause (g) of 6.7A and 6.7F(0) we have $\mathfrak{a}_1 \cap \mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_i: i < \alpha\}) \subseteq \bigcup_{\theta\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\theta[\bar{\mathfrak{a}}]$. So for $\alpha < \beta < |\mathfrak{a}|^+$, $\mathfrak{d}_\alpha \subseteq \mathfrak{a}_1 \cap \mathrm{pcf}_{\theta-\mathrm{complete}}\{\lambda_i: i < \alpha\} \subseteq \mathfrak{a}_1 \cap \mathrm{pcf}_{\theta-\mathrm{complete}}\{\lambda_i: i < \beta\} \subseteq \bigcup_{\theta\in\mathfrak{d}_\beta} \mathfrak{b}^1_\theta[\bar{\mathfrak{a}}]$. As the sequence is smooth (i.e. clause (f) of 6.7A) clearly $\alpha < \beta \Rightarrow \bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \subseteq \bigcup_{\mu\in\mathfrak{d}_\beta} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}]$.

So $\langle \bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}: \alpha < |\mathfrak{a}|^+\rangle$ is a non-decreasing sequence of subsets of $\mathfrak{a}$ of length $|\mathfrak{a}|^+$, hence for some $\alpha(*) < |\mathfrak{a}|^+$ we have:

$(*)_1$ $\alpha(*) \le \alpha < |\mathfrak{a}|^+ \Rightarrow \bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a} = \bigcup_{\mu\in\mathfrak{d}_{\alpha(*)}} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}$.

If $\tau \in \mathfrak{a}_1 \cap \mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_i: i < \alpha\})$ then $\tau \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a})$ (by 6.7F(2),(3)), and $\tau \in \mathfrak{b}^1_{\mu_\tau}[\bar{\mathfrak{a}}]$ for some $\mu_\tau \in \mathfrak{d}_\alpha$ so $\mathfrak{b}^1_\tau[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^1_{\mu_\tau}[\bar{\mathfrak{a}}]$, also $\tau \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{b}^1_\tau[\bar{\mathfrak{a}}] \cap \mathfrak{a})$ (by clause (e) of 6.7A), hence

$$\begin{aligned}\tau \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{b}^1_\tau[\bar{\mathfrak{a}}] \cap \mathfrak{a}) &\subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{b}^1_{\mu_\tau}[\bar{\mathfrak{a}}] \cap \mathfrak{a})\\ &\subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}\left(\bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}\right).\end{aligned}$$

So $\mathfrak{a}_1 \cap \mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_i: i < \alpha\}) \subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}\left(\bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}\right)$. But for each $\alpha < |\mathfrak{a}|^+$ we have $\lambda_\alpha > \sup \mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_i: i < \alpha\})$, whereas $\mathfrak{d}_\alpha \subseteq \mathrm{pcf}_{\sigma-\mathrm{complete}}\{\lambda_i: i < \alpha\}$, hence $\lambda_\alpha > \sup \mathfrak{d}_\alpha$ hence

$(*)_2$ $\lambda_\alpha > \sup_{\mu\in\mathfrak{d}_\alpha} \max\mathrm{pcf}\, \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \ge \sup \mathrm{pcf}_{\theta-\mathrm{complete}}\left(\bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}\right)$.

On the other hand,

$(*)_3$ $\lambda_\alpha \in \mathrm{pcf}_{\theta-\mathrm{complete}}\{\lambda_i: i < \alpha+1\} \subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}\left(\bigcup_{\mu\in\mathfrak{d}_{\alpha+1}} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}\right)$.

For $\alpha = \alpha(*)$ we get contradiction by $(*)_1 + (*)_2 + (*)_3$.

(5) Assume $\mathfrak{a}$, $\mathfrak{c}$, λ form a counterexample with λ minimal. Without loss of generality $|\mathfrak{a}|^{+3} < \mathrm{Min}(\mathfrak{a})$ and $\lambda = \max\mathrm{pcf}\,\mathfrak{a}$ and $\lambda = \max\mathrm{pcf}\,\mathfrak{c}$ (just let $\mathfrak{a}' =: \mathfrak{b}_\lambda[\mathfrak{a}]$, $\mathfrak{c}' =: \mathfrak{c} \cap \mathrm{pcf}_\theta[\mathfrak{a}']$; if $\lambda \notin \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c}')$ then necessarily $\lambda \in \mathrm{pcf}(\mathfrak{c}\backslash\mathfrak{c}')$ (by 6.7F(0)) and similarly $\mathfrak{c}\backslash\mathfrak{c}' \subseteq \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a}\backslash\mathfrak{a}')$ hence by 6.7F(2),(3) $\lambda \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a}\backslash\mathfrak{a}')$,
contradiction).

Also without loss of generality $\lambda \notin \mathfrak{c}$. Let κ, σ, $\bar{N}$, $\langle i_\alpha = i(\alpha): \alpha \le \sigma\rangle$, $\bar{\mathfrak{a}} = \langle \mathfrak{a}_i: i \le \sigma\rangle$ be as in 6.7A with $\mathfrak{a} \in N_0$, $\mathfrak{c} \in N_0$, $\lambda \in N_0$, $\sigma = |\mathfrak{a}|^+$, $\kappa = |\mathfrak{a}|^{+3} < \mathrm{Min}\,\mathfrak{a}$. We choose by induction on $\epsilon < |\mathfrak{a}|^+$, λ_ϵ, $\mathfrak{d}_\epsilon$ such that:

(a) $\lambda_\epsilon \in \mathfrak{a}_{\omega^2\epsilon+\omega+3}$, $\mathfrak{d}_\epsilon \in N_{i(\omega^2\epsilon+\omega+1)}$,

(b) $\lambda_\epsilon \in \mathfrak{c}$,

(c) $\mathfrak{d}_\epsilon \subseteq \mathfrak{a}_{\omega^2\epsilon+\omega+1} \cap \mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_\zeta: \zeta < \epsilon\})$,

(d) $|\mathfrak{d}_\epsilon| < \theta$,

(e) $\{\lambda_\zeta\colon \zeta < \epsilon\} \subseteq \bigcup_{\theta\in\mathfrak{d}_\epsilon} \mathfrak{b}_\theta^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$,

(f) $\lambda_\epsilon \notin \mathrm{pcf}_{\theta-\mathrm{complete}}\left(\bigcup_{\theta\in\mathfrak{d}_\epsilon} \mathfrak{b}_\theta^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]\right)$.

For every $\epsilon < |\mathfrak{a}|^+$ we first choose $\mathfrak{d}_\epsilon$ as the $<^*_\chi$-first element satisfying (c) + (d) + (e) and then if possible λ_ϵ as the $<^*_\chi$-first element satisfying (b) + (f). It is easy to check the requirements and in fact $\langle\lambda_\zeta\colon \zeta < \epsilon\rangle \in N_{\omega^2\epsilon+1}$, $\langle\mathfrak{d}_\zeta\colon \zeta < \epsilon\rangle \in N_{\omega^2\epsilon+1}$ (so clause (a) will hold). But why can we choose at all? Now $\lambda \notin \mathrm{pcf}_{\theta-\mathrm{complete}}\{\lambda_\zeta\colon \zeta < \epsilon\}$ as $\mathfrak{a}$, $\mathfrak{c}$, λ form a counterexample with λ minimal and $\epsilon < |\mathfrak{a}|^+$ (by 6.7F(3)). As $\lambda = \max\mathrm{pcf}\,\mathfrak{a}$ necessarily $\mathrm{pcf}_{\theta-\mathrm{complete}}(\{\lambda_\zeta\colon \zeta < \epsilon\}) \subseteq \lambda$ hence $\mathfrak{d}_\epsilon \subseteq \lambda$ (by clause (c)). By part (0) of the claim (and clause (a)) we know:

$$\begin{aligned}\mathrm{pcf}_{\theta-\mathrm{complete}}\left[\bigcup_{\mu\in\mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]\right] &= \bigcup_{\mu\in\mathfrak{d}_\epsilon} \mathrm{pcf}_{\theta-\mathrm{complete}}\left[\mathfrak{b}_\mu^{\omega^2+\omega+1}[\bar{\mathfrak{a}}]\right]\\ &\subseteq \bigcup_{\mu\in\mathfrak{d}_\epsilon}(\mu+1) \subseteq \lambda\end{aligned}$$

(note $\mu = \max\mathrm{pcf}\,\mathfrak{b}_\mu^\beta[\bar{\mathfrak{a}}]$). So $\lambda \notin \mathrm{pcf}_{\theta-\mathrm{complete}}\left(\bigcup_{\mu\in\mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]\right)$ hence by part (0) of the claim $\mathfrak{c} \not\subseteq \bigcup_{\mu\in\mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ so λ_ϵ exists. Now $\mathfrak{d}_\epsilon$ exists by 6.7A clause (h)$^+$.

Now clearly $\left\langle \mathfrak{a} \cap \bigcup_{\mu\in\mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]\colon \epsilon < |\mathfrak{a}|^+\right\rangle$ is non-decreasing (as in the earlier proof) hence eventually constant, say for $\epsilon \geq \epsilon(*)$ (where $\epsilon(*) < |\mathfrak{a}|^+$). But

(α) $\lambda_\epsilon \in \bigcup_{\mu\in\mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ [clause (e) in the choice of $\lambda_\epsilon, \mathfrak{d}_\epsilon$],

(β) $\mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] \subseteq \bigcup_{\mu\in\mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ [by clause (f) of 6.7A and (α) alone],

(γ) $\lambda_\epsilon \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{a})$ [as $\lambda_\epsilon \in \mathfrak{c}$ and a hypothesis],

(δ) $\lambda_\epsilon \in \mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}])$ [by (γ) above and clause (e) of 6.7A],

(ϵ) $\lambda_\epsilon \notin \mathrm{pcf}(\mathfrak{a} \setminus \mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1})$,

(ζ) $\lambda_\epsilon \in \mathrm{pcf}_{\theta-\mathrm{complete}}\left(\mathfrak{a} \cap \bigcup_{\mu\in\mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]\right)$ [by (δ) + (ϵ) + (β)].

But for $\epsilon = \epsilon(*)$, the statement (ζ) contradicts the choice of $\epsilon(*)$ and clause (f) above. $\blacksquare_{6.7\mathrm{F}}$

References

[ARSh153] U. Abraham, M. Rubin and S. Shelah, *On the consistency of some partition theorems for continuous colorings and the structure of $\aleph_1$-dense real order type*, Annals of Pure and Applied Logic **29** (1985), 123–206.

[BoSh210] R. Bonnet and S. Shelah, *Narrow Boolean algebras*, Annals of Pure and Applied Logic **28** (1985), 1–12.

[CEG] F. S. Cater, Paul Erdös and Fred Galvin, *On the density of λ-box products*, General Topology and its Applications **9** (1978), 307–312.

[CN1] W. W. Comfort and S. Negrepontis, *On families of large oscillation*, Fundamenta Mathematicae **75** (1972), 275–290.

[CN2] W. W. Comfort and S. Negrepontis, *The Theory of Ultrafilters, Grundlehren der Mathematischen Wissenschaften* **211**, Springer-Verlag, Berlin–Heidelberg–New York, 1974.

[CR] W. W. Comfort and Lewis C. Robertson, *Cardinality constraints for pseudocompact and for totally dense subgroups of compact topological groups*, Pacific Journal of Mathematics **119** (1985), 265–285.

[DoJe] T. Dodd and R. Jensen, *The Covering Lemma for K*, Annals of Mathematical Logic **22** (1982), 1–30.

[EK] R. Engelking and M. Karlowicz, *Some theorems of set theory and their topological consequences*, Fundamenta Math **57** (1965), 275–285.

[H] Edwin Hewitt, *A remark on density characters*, Bulletin of the American Mathematical Society **52** (1946), 641–643.

[Ma] Edward Marczewski, *Séparabilité et multiplication cartésienne des espaces topologiques*, Fundamenta Mathematicae **34** (1947), 127–143.

[Mo] D. Monk, *Cardinal functions of Boolean algebras*, circulated notes.

[P] E. S. Pondiczery, *Power problems in abstract spaces*, Duke Mathematical Journal **11** (1944), 835–837.

[Sh-b] S. Shelah, *Proper Forcing*, Lectures Notes in Mathematics **940**, Springer-Verlag, Berlin–New York, 1982.

[Sh-f] S. Shelah, *Proper and Improper Forcing*, Perspectives in Mathematical Logic, to appear.

[Sh-g] S. Shelah, *Cardinal Arithmetic*, Oxford University Press, 1994.

[Sh88] S. Shelah, *Classification theory for non-elementary classes II*, in *Abstract Elementary Classes*, Proc. U.S.A–Israel Conference on Classification Theory, Chicago 12/85 (J. T. Baldwin, ed.), Springer-Verlag Lecture Notes **1292** (1987), 419–497.

[Sh92] S. Shelah, *Remarks on Boolean algebras*, Algebra Universalis **11** (1980), 77–89.

[Sh93] S. Shelah, *Simple unstable theories*, Annals of Mathematical Logic **19** (1980), 177–203.

[Sh107] S. Shelah, *Models with second order properties IV, a general method and eliminating diamonds*, Annals of Mathematical Logic **25** (1983), 183–212.

[Sh111] S. Shelah, *On powers of singular cardinals*, Notre Dame Journal of Formal Logic **27** (1986), 263–299.

[RuSh117] M. Rubin and S. Shelah, *Combinatorial problems on trees: partitions, Δ-systems and large free subtrees*, Annals of Pure and Applied Logic **33** (1987), 43–81.

[ShLH162] S. Shelah, C. Laflamme and B. Hart, *Models with second order properties V : a general principle*, Annals of Pure and Applied Logic **64** (1993), 169–194.

[Sh233] S. Shelah, *Remarks on the number of ideals of Boolean algebra and open sets of a topology*, Springer-Verlag Lecture Notes **1182** (1986), 151–187.

[Sh282] S. Shelah, *Successors of singulars, cofinalities of reduced products of cardinals and productivity of chain conditions*, Israel Journal of Mathematics **62** (1988), 213–256.

[Sh288] S. Shelah, *Strong partition relations below the power set: Consistency, was Sierpinski right, II?*, Proceedings of the Conference on Set Theory and its Applications in honor of A. Hajnal and V. T. Sos, Budapest, 1/91, Colloquia Mathematica Societatis Janos Bolyai, Sets, Graphs and Numbers, Vol. 60, 1991.

[Sh300] S. Shelah, *Universal classes*, Chapter I–IV, Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985 (J. Baldwin, ed.), Springer-Verlag Lecture Notes **1292** (1987), 264–418.

[Sh326] S. Shelah, *Viva la difference I: Nonisomorphism of ultrapowers of countable models*, in *Set Theory of the Continuum*, Mathematical Sciences Research Institute Publications, Vol. 26, Springer-Verlag, Berlin, 1992, pp. 357–405.

[GiSh344] M. Gitik and S. Shelah, *On certain indestructibility of strong cardinals and a question of Hajnal*, Archive for Mathematical Logic **28** (1989), 35–42.

[Sh345] S. Shelah, *Products of regular cardinals and cardinal invariants of products of Boolean algebras*, Israel Journal of Mathematics **70** (1990), 129–187.

[Sh345a] S. Shelah, *Basic: Cofinalities of small reduced products*, in*Cardinal Arithmetic*, Chapter I, Oxford University Press, 1994.

[Sh345b] S. Shelah, *Entangled orders and narrow Boolean algebras*, in *Cardinal Arithmetic*, Appendix 2, Oxford University Press, 1994.

[Sh355] S. Shelah, *$\aleph_{\omega+1}$ has a Jonsson algebra*, in *Cardinal Arithmetic*, Chapter II, Oxford University Press, 1994.

[Sh365] S. Shelah, *There are Jonsson algebras in many inaccessible cardinals*, in *Cardinal Arithmetic*, Chapter III, Oxford University Press, 1994.

[Sh371] S. Shelah, *Advanced: cofinalities of small reduced products*, in *Cardinal Arithmetic*, Chapter VIII, Oxford University Press, 1994.

[Sh386] S. Shelah, *Bounding pp(μ) when* $\mathrm{cf}(\mu) > \mu > \aleph_0$ *using ranks and normal ideals*, in *Cardinal Arithmetic*, Chapter V, Oxford University Press, 1994.

[Sh400] S. Shelah, *Cardinal arithmetic*, in *Cardinal Arithmetic*, Chapter IX, Oxford University Press, 1994.

[Sh410] S. Shelah, *More on cardinal arithmetic*, Archive for Mathematical Logic **32** (1993), 399–428.

[GiSh412] M. Gitik and S. Shelah, *More on ideals with simple forcing notions*, Annals of Pure and Applied Logic **59** (1993), 219–238.

[Sh420] S. Shelah, *Advances in cardinal arithmetic*, Proceedings of the Banff Conference in Alberta; 4/91, *Finite and Infinite Combinatorics in Sets and Logic* (N. W. Saure et al., eds.), Kluwer Academic Publishers, Dordrecht 1993, pp. 355–383.

[Sh460] S. Shelah, *The Generalized Continuum Hypothesis revisited*, submitted to Israel Journal of Mathematics.

[Sh513] S. Shelah, *PCF and infinite free subsets*, submitted to Archive for Mathematical Logic.

[Sh589] S. Shelah, *PCF theory: applications*, in preparation.